Lecture 1 – 21.04.2020 – 12:15 via Zoom

Schedule: Thusday 12.15-13.45 and Thursday 12.15-13.45, Online until further notice.

Tutorial classes: Daria Frolova (Wed 16-18, SemR 1.007), Min Liu (Monday 16-18, SemR 1.007) Online until further notice.

Handling of sheets via **eCampus**. Upload the sheet when you completed it, and download there the corrections. *Sheet must be handled in LaTeX.*

Subscribe also to the Exercise in Stochastic Analsysis in eCampus

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Prerequisites

"Foundations/Introduction on Stochastic Analysis". Probability measures, continuous time stochastic processes, Kolmogorov's construction of stoch. proc., continuous time martingales, stochastic integration, Ito formula, SDE. Give a look at

https://www.iam.uni-bonn.de/abteilung-gubinelli/teaching/found-stoch-analysis-ws1920/

Introduction

Stochastic Analysis: set of tools to study stochastic (continuous) processes (i.e. Brownian motion, semimartingales, solutions to SDE, random fields).

Wiener '40 (Brown. mot., Lebesgue's theory)

Doob's/Levy/Ito ('40-'50) / Kunita/Watanabe/McKean/Malliavin/…

Malliavin derivative / White-noise calculus

Generalisation of analysis adapted to the study of stoch. proc.

Content of the course

- **Stoch. Diff. equations**: weak, strong, martingale problems. Links between the various notions. Including questions of uniqueness of solutions (pathwise uniq, weak uniq, uniq. of mart. problem).

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s.$$

- **Techniques for SDEs**. time-change ($X_t = Y_{f(t)}$), Girsanov's theorem ($\mathbb{Q} \ll \mathbb{P}$, $(\mathscr{F}_t)_t$), Tanaka's formula, conditioning (Doob's h-transform), singular conditioning (cond. on events of prob. zero). Doss–Sussmann technique (exact solutions to SDEs, link with control theory and ODE theory). Relation with PDE theory.

- **Martingale representation theorem.** (every mart. on a Brownian filtration is a stoch. integral). The formula of Boué–Dupuis ('90) - gives a variational formula for expectation values over a Brownian filtration. Large deviations for SDE:

$$\mathrm{d}X_t^\varepsilon = b(X_t^\varepsilon)\mathrm{d}t + \varepsilon\sigma(X_t^\varepsilon)\mathrm{d}B_t$$

$\varepsilon > 0$ small. $(X^\varepsilon)_{\varepsilon \geqslant 0}$ What happens for $\mu^\varepsilon(A) := \mathbb{P}(X^\varepsilon \in A)$ as $\varepsilon \to 0$. $\mu^\varepsilon \to \delta_{\mathrm{ODE}}$. How fast is a question for large deviations theory.

$$\mu^\varepsilon(A) \approx \exp\left(-\frac{I(A)}{\varepsilon^2}\right), \qquad I(A) = \inf_{f \in A} I(f).$$

- **Diffusions on manifolds.** $(X_t \in \mathscr{M})_{t \geqslant 0}$ SDE??? Brownian motion on $\mathscr{M}$, relation with differential geometry. Δ Laplace–Beltrami.

- **Numerical methods for SDE.** $(X_t^n)_{t \geqslant 0}$ Euler-Maruyama method. Strong, weak approximations. As $n \to \infty$,

$$\mathbb{E}(f(X_t)) \approx \mathbb{E}(f(X_t^n)), \qquad \mathbb{E}\|X_t - X_t^n\| \approx 0.$$

Stochastic Taylor expansion (iterated stochastic integrals)

$$f(B_t) = f(B_s) + f'(B_s)(B_t - B_s) + f''(B_s)\underbrace{\int_s^t \left(\int_s^u \mathrm{d}B_v\right)\mathrm{d}B_u}_{\mathbb{B}_{s,t}^2} + \cdots$$

- **Rough path theory** (?) (robust and path-wise intergration theory for irregular processes) (T. Lyons '98)

- **Malliavin calculus** (?) (P. Malliavin '80) Analysis on infinite dimensional measure spaces. Wiener measure $\mathscr{W}(A) = \mathbb{P}(B \in A)$ $A \in \mathscr{B}(C([0,1];\mathbb{R}))$. $\mathscr{W}$ is a probability measure on $C([0,1];\mathbb{R})$. Wiener measure is a *replacement* for Lebesgue measure in $C([0,1];\mathbb{R})$. Quasi-invariant under shift. Lebesgue/Sobolev type spaces on $C([0,1];\mathbb{R})$. Notion of derivative: Malliavin derivative. Link to the martingale rep. theorem and to iterated stochastic integrals.

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1 Stochastic differential equations

Setting. Probability space $(\Omega, \mathscr{F}, \mathbb{P})$, filtration $(\mathscr{F}_t)_{t \geqslant 0}$ right-continuous, completed.

Definition 1. *A weak solution of the SDE in* $\mathbb{R}^n$

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t \in [0, T]$$

$$X_0 = x \in \mathbb{R}^n$$

is a pair of adapted processes (X, B) *where* $(B_t)_{t \geqslant 0}$ *is a m-dimensional Brownian motion and* b, σ *are coefficients* $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m; \mathbb{R}^n)$ *such that almost surely*

$$\int_0^t |b(X_s)| \mathrm{d}s < \infty, \quad \int_0^t \mathrm{Tr}(\sigma(X_s)\sigma(X_s)^T) \mathrm{d}s < \infty, \qquad t \in [0, T]$$

and that

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s, \qquad t \in [0, T].$$

$\sigma = (\sigma_\alpha)_{\alpha = 1, \ldots, m}$ family of vector-fields $\sigma_\alpha: \mathbb{R}^n \to \mathbb{R}^n$ (this is the right point of view on manifolds)

Control-theory point of view:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sum_{\alpha = 1}^m \sigma_\alpha(X_t)\mathrm{d}B_t^\alpha.$$

$$\sum_{\alpha = 1}^m \int_0^t |\sigma_\alpha(X_t)|^2 \mathrm{d}s < \infty.$$

Definition 2. *A strong solution to the SDE above is a weak solution such that X is adapted to the filtration* $(\mathscr{F}_t^B)_{t \geqslant 0}$ *generated by B,* $\mathscr{F}_t^B := \overline{\sigma(B_s : s \in [0, t])}$.

$$X_t \hat{\in} \mathscr{F}_t \Rightarrow X_t(\omega) = \Phi_t((B_s(\omega))_{s \in \in [0, t]})$$

$\Phi_t: C([0, t]; \mathbb{R}^m) \to \mathbb{R}^n$. While in general we could have

$$X_t(\omega) = \Phi_t((B_s(\omega))_{s \in \in [0, t]}, N(\omega)).$$

Facts.

- There are weak solutions which are not strong. (Tanaka's example)
- There are SDEs which do not have strong solutions.
- A weak solution is really the data $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t \geqslant 0}, X, B)$.

Definition 3. *An SDE has* ***uniqueness in law*** *iff two solutions* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t \geqslant 0}, X, B)$ $(\Omega', \mathscr{F}', \mathbb{P}', (\mathscr{F}_t')_{t \geqslant 0}, X', B')$ *are such that*

$$\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{P}'}(X').$$

Definition 4. *An SDE has **pathwise uniqueness** if for any two solutions X, X' defined on the same filt. prob. space and with the **same** BM B we have that they are indistinguishable, i.e.*

$$\mathbb{P}(\exists t \in [0,T] : X_t \neq X'_t) = 0.$$