SDE techniques: Doob's transform

Let $(X_t, B_t)_{t \geqslant 0}$ be the solution of an SDE with Markovian drift $b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and diffusion coefficient σ: $\mathbb{R}_+ \times \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$ where B is the Brownian motion driving the SDE.

Let $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_{>0})$ be a stricly positive function such that

$$(\partial_t + \mathscr{L}) h(t, x) = 0,$$

for all $t \in [0, t_*]$ and $x \in \mathbb{R}^n$ where $\mathscr{L}$ is the generator of the SDE, i.e. $\mathscr{L} = b \cdot \nabla + \frac{1}{2} \mathrm{Tr}[\sigma \sigma^T \nabla^2] \big)$.

By Ito formula the process $Z_t := h(t, X_t)$ is a positive local martingale. Let us assume that $(Z_t)_{t \in [0, t_*]}$ is a (true) martingale and that $Z_0 = h(0, X_0) = 1$ (this can be always arranged by normalizing h). Then we can use the process $(Z_t)_t$ to define a new measure

$$\mathrm{d}\mathbb{Q} := Z_{t_*} \mathrm{d}\mathbb{P}.$$

(If needed we can extend $Z_t = Z_{t_*}$ if $t > t_*$). Note that by construction the process Z is continuous and $Z_0 = 1$. By using Girsanov's theorem we know that the process

$$\tilde{B} = B - [B, L]$$

is a $\mathbb{Q}$-Brownian motion where L is the only local martingale such that $Z = \mathscr{E}(L)$. Since $\mathrm{d}Z_t = Z_t \mathrm{d}L_t$ we have that

$$\mathrm{d}Z_t = \sigma(t, X_t)^T \nabla h(t, X_t) \cdot \mathrm{d}B_t, \qquad \mathrm{d}L_t = Z_t^{-1} \mathrm{d}Z_t = \frac{\sigma^T(t, X_t) \nabla h(t, X_t)}{h(t, X_t)} \cdot \mathrm{d}B_t = \sigma^T(t, X_t) \nabla \log h(t, X_t) \cdot \mathrm{d}B_t$$

for $t \leqslant t_*$ and $\mathrm{d}Z_t = 0$ if $t > t_*$. Therefore

$$\mathrm{d}\tilde{B}_t = \mathrm{d}B_t - \sigma^T(t, X_t) \nabla \log h(t, X_t) \mathrm{d}t, \qquad t \in [0, t_*],$$

and $\mathrm{d}\tilde{B}_t = \mathrm{d}B_t$ if $t > t_*$. As consequence the process X solves now a new SDE (under $\mathbb{Q}$)

$$\mathrm{d}X_t = \underbrace{[b(t, X_t) + \sigma(t, X_t) \sigma^T(t, X_t) \nabla \log h(t, X_t)] \mathrm{d}t}_{\tilde{b}(t, X_t) \mathrm{d}t} + \sigma(t, X_t) \mathrm{d}\tilde{B}_t, \qquad t \in [0, t_*]$$

with the same diffusion coeffient σ but a new drift

$$\tilde{b}(t, x) = b(t, x) + \mathbb{1}_{t \in [0, t_*]} (\sigma \sigma^T \nabla \log h)(t, x), \qquad t \geqslant 0, x \in \mathbb{R}^n.$$

This construction is called *Doob's h-transform*.

Exercise 1. Try to perform the same construction for a martingale problem, i.e. not relying on the process B but only on X. I.e. starting from a measure $\mathbb{P}$ on the canonical path space $C(\mathbb{R}_+; \mathbb{R}^n)$ solving the martingale problem for $\mathscr{L}$ construct a new measure $\mathbb{Q}$ which solves a new martingale problem with a modified drift as above.

Example 1. Take $h(t, x) = \exp\left(\gamma \cdot x - \frac{1}{2} |\gamma|^2\right)$ where $\gamma \in \mathbb{R}^n$ and $t \geqslant 0$. Then the Doob's h-transformed process of a Brownian motion with this funciton gives a Brownian motion with drift.

If $(Z_t)_t$ is only a martingale in an open interval $I = [0, t_*)$ with possibly $t_* = +\infty$. Then we can still define $\mathbb{Q}$ on $\mathscr{F}_t$ to be given by $\mathrm{d}\mathbb{Q}|_{\mathscr{F}_t} := Z_t \mathrm{d}\mathbb{P}|_{\mathscr{F}_t}$ and check that this gives a well-defined probability measure on $\mathscr{F}_\infty = \vee_{t \geqslant 0} \mathscr{F}_t$. In this case is natural to restict all the measures to $\mathscr{F}_\infty$ i.e. to require $\mathscr{F}_\infty = \mathscr{F}$.

Remark 2. We do not need to require that h is positive everywhere (actually this will not be the case in the applications). What we need is that the process $Z_t = h(t, X_t)$ is a local martingale, i.e. $(\partial_t + \mathscr{L}) h(t, X_t) = 0$ a.s. and for almost every $t \geqslant 0$ and that $Z_t > 0$ almost surely. If h is not stricly positive we can always define the stopping time $T = \inf \{t \geqslant 0 : Z_t = 0\}$, then the stopped process $(Z_t^T)_{t \geqslant 0}$ is a positive local martingale and some condition is needed to ensure that it is a martingale. Remember that we require that $Z_0 = 1$ and by construction $(Z_t)_{t \geqslant 0}$ is continuous. In this setting one can perform the Doob's transform up to the random stopping time T. Note that under the measure $\mathbb{Q}$ we always have $T = +\infty$ almost surely.

1 Diffusion bridges

We use now Doob's transform to describe the regular conditional law of a Markovian diffusion $(X_t)_{t \geqslant 0}$ conditioned on the event that $X_T = y$ with $T > 0$ and deterministic, and $y \in \mathbb{R}^n$. I will assume also that $X_0 = x_0$. We need to assume that the process $(X_t)_{t \geqslant 0}$ is a Markov process with transition density given by

$$\mathbb{P}(X_t \in \mathrm{d}x' | X_s = x) = p(s, x; t, x') \mathrm{d}x', \qquad s < t \in [0, T], x, x' \in \mathbb{R}^n,$$

for some measurable and positive function p. Note that we cannot take $s = t$ here. Recall that $\mathbb{P}(X_t \in \mathrm{d}y | X_s = x)$ means the regular conditional probability kernel for the conditional law of X_t given X_s.

Define now the function

$$h^y(s, x) := \frac{p(s, x; T, y)}{p(0, x_0; T, y)}, \qquad s \in [0, T), x \in \mathbb{R}^n.$$

Let $Z_t^y := h^y(t, X_t)$, this is non-negative process, and it is also a martingale, indeed by the Markov property of X

$$\mathbb{E}[Z_t^y | \mathscr{F}_s] = \mathbb{E}[h^y(t, X_t) | \mathscr{F}_s] = \mathbb{E}[h^y(t, X_t) | X_s] = \int_{\mathbb{R}^n} h^y(t, x') p(s, X_s; t, x') \mathrm{d}x'$$

$$= \frac{1}{p(0, x_0; T, y)} \int_{\mathbb{R}^n} p(s, X_s; t, x') p(t, x'; T, y) \mathrm{d}x' = \frac{p(s, X_s; T, y)}{p(0, x_0; T, y)} = Z_s^y$$

by Chapman–Kolmogorov equations (the consistency condition for the transition density of a Markov process).

We want to define a probability kernel $(\mathbb{Q}^y)_{y \in \mathbb{R}^n}$ on $(\Omega, \mathscr{F})$ such that they are the regular conditional probabiliy of $\mathbb{P}$ given X_T, that is they have to satisfy

$$\mathbb{P}(A) = \mathbb{E}[\mathbb{P}(A | X_T)] = \mathbb{E}[\mathbb{Q}^{X_T}(A)] = \int_{\mathbb{R}^n} \mathbb{Q}^y(A) \mathbb{P}(X_T \in \mathrm{d}y) = \int_{\mathbb{R}^n} \mathbb{Q}^y(A) p(0, x_0; T, y) \mathrm{d}y$$

for all $A \in \mathscr{F}$. Take $A \in \mathscr{F}_s$ for some $s < T$, by Markov property we have for any bounded measurable g,

$$\mathbb{E}[\mathbb{1}_A g(X_T)] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[g(X_T) | \mathscr{F}_s]] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[g(X_T) | X_s]] = \mathbb{E}\left[\mathbb{1}_A \int_{\mathbb{R}^n} g(y) p(s, X_s; T, y) \mathrm{d}y\right]$$

$$= \int_{\mathbb{R}^n} g(y) \mathbb{E}[\mathbb{1}_A p(s, X_s; T, y)] \mathrm{d}y$$

since

$$\mathbb{E}[g(X_T) | X_s] = \int_{\mathbb{R}^n} g(y) p(s, X_s; T, y) \mathrm{d}y.$$

This means that we have $\mathbb{P}(A | X_T) = q(X_T)$ and we can take

$$q(y) := \mathbb{E}\left[\mathbb{1}_A \frac{p(s, X_s; T, y)}{p(0, x_0; T, y)}\right],$$

since we have proven that

$$\mathbb{E}[q(X_T) g(X_T)] = \mathbb{E}[\mathbb{1}_A g(X_T)] = \int_{\mathbb{R}^n} g(y) q(y)\, p(0, x_0; T, y) \mathrm{d}y.$$

As a consequence we can take

$$\mathbb{Q}^y(A) := \mathbb{E}\left[\mathbb{1}_A \frac{p(s,X_s;T,y)}{p(0,x_0;T,y)}\right], \qquad A \in \mathscr{F}_s$$

and have that $y \mapsto \mathbb{Q}^y$ indentify a well-defined probability kernel on $\mathscr{F}_{T-}$ since for any $A \in \mathscr{F}_{T-}$ the function $y \mapsto \mathbb{Q}^y(A)$ is measurable in y and for any y, $\mathbb{Q}^y$ is a probability in A.

Remark 3. Is it possible with some care to extend $\mathbb{Q}^y$ to the full $\mathscr{F}$, but we refrain to do so here.

We have now the formula

$$\mathbb{P}(A|X_T) = \mathbb{Q}^{X_T}(A), \qquad A \in \mathscr{F}_{T-}.$$

I want now to describe better the measure $\mathbb{Q}^y$ (at least up to time T), we observe that $\mathbb{Q}^y$ is obtained as the Doob's h-transform of $\mathbb{P}$ in the interval $[0,T)$ with $h = h^y$ function

$$h^y(s,x) := \frac{p(s,x;T,y)}{p(0,x_0;T,y)}, \qquad s \in [0,T), x \in \mathbb{R}^n.$$

As a consequence we can show that the process X under $\mathbb{Q}^y$ satisfies an SDE provided I can apply Ito formula to h^y, that is I have to require that $(s,x) \mapsto p(s,x;T,y)$ is $C^{1,2}([0,T) \times \mathbb{R}^n)$. Given that Doob's transform give that X under $\mathbb{Q}^y$ solves the new SDE (or an equivalent martingale problem)

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma\sigma^T \nabla \log h^y(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t, \qquad t \in [0,T).$$

Is easy to see from specific examples that the function $\sigma\sigma^T \nabla \log h^y(t,x)$ is singular when $t \nearrow T$.

Exercise 2. Compute the SDE satisfied by a n-dimensional Brownian motion when we condition it to reach the point y at time $T > 0$.

Observe that under $\mathbb{Q}^y$ we have that

$$\mathbb{Q}^y\left(\lim_{t \uparrow T} X_t = z\right) = \mathbb{1}_{z=y}.$$

for any $y, z \in \mathbb{R}^n$. Observe also that

$$\mathbb{P}\left(\lim_{t \uparrow T} X_t = y\right) = \mathbb{P}(X_T = y) = 0$$

since X_T has density $p(0,x_0;T,\cdot)$. So the measures $\mathbb{Q}^y$ are all singular wrt. $\mathbb{P}$.

Next week: more complex conditionings, e.g. diffusion condioned never to leave a given region.
