Lecture 11 – 2020.05.26 – 12:15 via Zoom

SDE techniques: Doob's transform/Conditioning

Last lecture: we described the law of a diffusion $(X_t)_{t\geqslant 0}$ conditioned to reach a point y at a time T.

More precisely, take $\mathbb{P}$ to be the law of a diffusion $(X_t)_{t\geqslant 0}$,. The goal was to identify the probability kernel $y\in\mathbb{R}^n\mapsto\mathbb{P}^y\in\Pi(\mathscr{C}^n)$ such that we can disintegrate $\mathbb{P}$ as

$$\mathbb{P}(A)=\int_{\mathbb{R}^n}\mathbb{P}^y(A)\mathbb{P}(X_T\in\mathrm{d}y)=\mathbb{E}[\mathbb{P}^{X_T}(A)],\qquad A\in\mathscr{F},\tag{1}$$

where $\mathbb{P}(X_T\in\mathrm{d}y)$ represent the law of X_T under $\mathbb{P}$. We define the law of X conditioned to reach a point y at a time T as the law of X under $\mathbb{P}^y$. Being the event $\{X_T=y\}$ of zero probability for $\mathbb{P}$ in general, this is a reasonable way to define this event. We see indeed that $\mathbb{P}^{X_T}(A)=\mathbb{E}[\mathbb{1}_A|X_T]$.

We had to assume that the process $(X_t)_{t\geqslant 0}$ is Markov wrt. the given filtration $(\mathscr{F}_t)_{t\geqslant 0}$ and that it has a transition probability given by the density $p(s,x;s',x')$ so that

$$\mathbb{P}(X_{s'}\in\mathrm{d}x'|X_s=x)=p(s,x;s',x')\mathrm{d}x',\qquad s<s',x,x'\in\mathbb{R}^n.$$

We can the introduce the *martingale* $Z_t^y=h^y(t,X_t)$ $t\in[0,T)$, given by

$$h^y(t,x)=\frac{p(t,x;T,y)}{p(0,x_0;T,y)},\qquad t\in[0,T),$$

where we assume that $X_0=x_0\in\mathbb{R}^n$ and that $p(t,x;T,y)>0$ for all x and $t\in[0,T)$. (this can be obtained by first conditioning on X_0 and then performing the construction). Usually $p(t,x;T,y)$ is not well defined when $t\to T$. E.g. in the case of Brownian motion one has

$$p(t,x;T,y)=(2\pi(T-t))^{-d/2}\exp\left(-\frac{|x-y|^2}{2(T-t)}\right).$$

Then one use this to construct the Doob's transformed measure $\mathbb{P}^y$ on $\mathscr{F}_{T-}$ by letting

$$\mathrm{d}\mathbb{P}^y|_{\mathscr{F}_t}=Z_t^y\mathrm{d}\mathbb{P}|_{\mathscr{F}_t},\quad t\in[0,T).$$

And one can check that this definition satisfy (1) for $A\in\mathscr{F}_{T-}$. Now if $A_2\in\sigma(X_t:t\geqslant T)$ we have

$$\mathbb{P}(A_2)=\mathbb{E}[\mathbb{E}[\mathbb{1}_{A_2}|\mathscr{F}_T]]=\mathbb{E}[\mathbb{E}[\mathbb{1}_{A_2}|X_T]]=\mathbb{E}[\varphi^{A_2}(X_T)]$$

with $\varphi^A(x)=\mathbb{E}[\mathbb{1}_A|X_T=x]$. So now consider also an event $A_1\in\sigma(X_t:t<T)$. In this case we have

$$\mathbb{P}(A_1\cap A_2)=\mathbb{E}[\mathbb{1}_{A_1}\mathbb{E}[\mathbb{1}_{A_2}|\mathscr{F}_T]]=\mathbb{E}[\mathbb{1}_{A_1}\varphi^{A_2}(X_T)]=\int_{\mathbb{R}^n}\mathbb{P}^y(A_1)\varphi^{A_2}(y)\mathbb{P}(X_T\in\mathrm{d}y).$$

Let assume that we have proven that

$$\mathbb{P}^y\left(\lim_{t\uparrow T}X_t=x\right)=\mathbb{1}_{x=y},\qquad x\in\mathbb{R}^n,$$

then we can write

$$\mathbb{P}(A_1\cap A_2)=\int_{\mathbb{R}^n}\underbrace{\mathbb{E}^y[\mathbb{1}_{A_1}\varphi^{A_2}(X_T)]}_{\mathbb{P}^y(A_1\cap A_2)}\mathbb{P}(X_T\in\mathrm{d}y).$$

So this shows us that we can define

$$\mathbb{P}^y(A_1\cap A_2)=\mathbb{E}^y[\mathbb{1}_{A_1}\varphi^{A_2}(X_T)]=\mathbb{E}^y[\mathbb{1}_{A_1}]\varphi^{A_2}(y).$$

This defines $\mathbb{P}^y$ in $\sigma(X_t : t < T) \vee \sigma(X_t : t \geqslant T) = \sigma(X_t : t \geqslant 0)$. So $\mathbb{P}^y$ can be used to define the conditional law of X.

One can then show that if the process X satisfies the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \qquad t \geqslant 0$$

then under $\mathbb{P}^y$ the process X satisfies the SDE (provided $h^y(t,x)$ is $C^{1,2}$ for any $t < T$)

$$\mathrm{d}X_t = [b(t, X_t) + (\sigma\sigma^T \nabla \log h^y)(t, X_t)]\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t < T$$

and

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t \geqslant T.$$

Recall that under $\mathbb{P}^y$ we have $X_{T-} = X_T = y$.

Remark 1. This approach can be exteded to condition a diffusion to reach a sequence of states $y_1, \ldots, y_n$ at given times $T_1 < \cdots < T_n$.

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1 Condition a diffusion to not leave a domain

Consider the following situation: we want to condition a one dimensional Brownian motion $(B_t)_{t \geqslant 0}$ to stay positive for all times $t \geqslant 0$. This event has probability zero (since eventually BM will visit zero and by strong Markov property will have $1/2$ probability to go negative + Borel-Cantelli). So the idea is to use less singular conditioning to arrive to describe this event.

Assume $B_0 = x_0 > 0$. In this case the convenient thing to do is to fix $R > x_0$ and ask consider the stopping time

$$T_R := \inf\{t \geqslant 0 : B_t \notin [0, R]\}$$

and the event $E_R := \{B_{T_R} = R\}$. Now we know that $\mathbb{P}(E_R) = x_0 / R \in (0,1)$. We can then define the conditional probability

$$\mathbb{P}^R(A) := \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)}$$

and now we would like to send $R \to \infty$ and study the limit.

Let $T_x = \inf\{t \geqslant 0 : B_t = x\}$. We want to say that

$$\{T_0 = +\infty\} = \cap_{R > 0}\{B_{T_R} = R\}$$

note that $(E_R)_R$ is a descreasing sequence of events.

I want to describe $\mathbb{P}^R$. Let $A \in \mathscr{F}_s$, observe that

$$\mathbb{P}^R(A) = \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)} = \frac{\mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{E_R} | \mathscr{F}_s]]}{\mathbb{P}(E_R)} = \frac{\mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{T_R > s} \mathbb{1}_{E_R} | \mathscr{F}_s]] + \mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{T_R \leqslant s} \mathbb{1}_{E_R} | \mathscr{F}_s]]}{\mathbb{P}(E_R)}$$

$$= \frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R > s} \mathbb{P}_{X_s}(E_R)] + \mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R \leqslant s} \mathbb{1}_{B_{T_R} = R}]}{\mathbb{P}(E_R)}$$

where $\mathbb{P}_x(E_R)$ is the probability of E_R for a BM starting at x at time 0. Note that $\mathbb{P}_0(E_R) = 0$ and $\mathbb{P}_R(E_R) = 1$ therefore setting

$$h(x) = \mathbb{P}_x(E_R) / \mathbb{P}_{x_0}(E_R) = x / x_0$$

we have that

$$\mathbb{P}^R(A) = \mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R > s} h(B_s)] + \mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R \leqslant s} h(B_{T_R})] = \mathbb{E}[\mathbb{1}_A h(B_{s \wedge T_R})].$$

Remember that we did this for any $s \geqslant 0$ and $A \in \mathscr{F}_s$. So

$$\mathrm{d}\mathbb{P}^R|_{\mathscr{F}_s} = h(B_{s \wedge T_R})\mathrm{d}\mathbb{P}|_{\mathscr{F}_s}.$$

If we take $Z_t^R = h(B_{t \wedge T_R})$ then $(Z_t^R)_{t \geqslant 0}$ is a non-negative martingale (indeed $0 \leqslant Z_t^R \leqslant R$).

We have to pay attention to the fact that Z_t^R could touch zero and this happens at the stopping time T_0. After time T_0 the process Z_t^R will stay in zero.

(Next lecture we continue)

Note that

$$Z_t^R = \mathscr{E}(L)_t, \qquad L_t = \int_0^t \mathbb{1}_{s \leqslant T_R} (\log h)'(B_s)\mathrm{d}B_s.$$

By Girsanov's theorem we have that under the measure $\mathbb{P}^R$ the process B satisfy the SDE

$$\mathrm{d}B_t = \frac{\mathbb{1}_{t \leqslant T_R}}{B_t}\mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0,$$

where W is a $\mathbb{P}^R$-Brownian motion.
