Lecture 12 – 2020.05.28 – 12:15 via Zoom

SDE techniques: Doob's transform/Conditioning (II)

1 Conditioning Brownian motion to stay positive

$(B_t)_{t\geqslant 0}$ BM. $R>0$ $T_x := \inf\{t\geqslant 0: B_t = x\}$ and $S_R := T_R \wedge T_0$. $B_0 = x_0 \in (0,R)$. Note that $\mathbb{P}(T_x < \infty) = 1$ for all $x\in\mathbb{R}$. We introduced the measure

$$\mathbb{Q}^R(A) = \mathbb{E}(\mathbb{1}_A h(B_{s\wedge S_R})), \qquad A\in\mathscr{F}_s$$

with $h(x) = x/x_0$. By Girsanov's theorem we have that under $\mathbb{Q}^R$ $(B_t)_{t\geqslant 0}$ solves the SDE

$$\mathrm{d}B_t = \frac{\mathbb{1}_{t<S_R}}{B_t}\mathrm{d}t + \mathrm{d}W_t, \qquad t\geqslant 0$$

where W is a $\mathbb{Q}^R$ Brownian motion. Important observation

$$\mathbb{Q}^R(T_0<T_R) = \mathbb{E}(\mathbb{1}_{T_0<T_R} h(B_{S_R})) = \mathbb{E}(\mathbb{1}_{T_0<T_R} h(B_{T_0})) = 0$$

so under $\mathbb{Q}^R$ we will never touch 0 before R and therefore under $\mathbb{Q}^R$ we have $S_R = T_R$ almost surely and that $B_t > 0$ for all $t\in[0,T_R]$.

Now we want to take the limit $R\to\infty$. Observe that if $A\in\mathscr{F}_{T_R}$ then $\mathbb{Q}^R(A) = \mathbb{Q}^{R'}(A)$ for any $R'\geqslant R$ since $T_{R'} > T_R$ and

$$\mathbb{E}_{\mathbb{Q}^{R'}}[\mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A h(B_{T_R\wedge S_{R'}})] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A \mathbb{E}[h(B_{T_R\wedge S_{R'}})|\mathscr{F}_{T_R}]] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A h(B_{T_R\wedge S_R})] = \mathbb{E}_{\mathbb{Q}^R}[\mathbb{1}_A].$$

Thefore for any $R>0$ and $A\in\mathscr{F}_{T_R}$ we can define a measure $\mathbb{Q}$ by letting $\mathbb{Q}(A) := \lim_{R'\to\infty}\mathbb{Q}^{R'}(A)$. Therefore the measure is well defined on $\vee_{R\geqslant 0}\mathscr{F}_{T_R}$. Observe that $\lim_{R\to\infty}\mathbb{Q}(T_R\leqslant s) = 0$ by continuity of B. For any $A\in\mathscr{F}_s$ we can define $\mathbb{Q}(A) := \lim_{R\to\infty}\mathbb{Q}(A, T_R > s)$ and check that it is the unique extension of $\mathbb{Q}$ which is consistent with it on $\vee_{R\geqslant 0}\mathscr{F}_{T_R}$.

Under $\mathbb{Q}$ the process B satisfies the SDE

$$\mathrm{d}B_t = \frac{1}{B_t}\mathrm{d}t + \mathrm{d}W_t, \qquad t\geqslant 0.$$

Therefore we discovered that under $\mathbb{Q}$ the process B satisfies the SDE defining the Bessel process R of dimension $d=3$:

$$\mathrm{d}R_t = \frac{d-1}{2}\frac{1}{R_t}\mathrm{d}t + \mathrm{d}W_t, \qquad t\geqslant 0.$$

Recall that the Bessel process $R_t = |X_t|$ is defined as the modulus of a d-dimensiona Brownian motion $(X_t)_{t\geqslant 0}$.

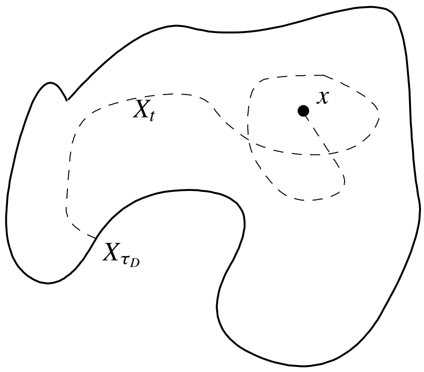
Theorem 1. *The law of a one dimensional Brownian motion conditioned never to hit zero is the same as the law of the modulus of a* 3d *Brownian motion.*

From this identification we can derive the corollary that the BM conditioned not to hit zero is transient and go to infinity with probability 1.

2 Condition a diffusion not to leave a domain

In the previous argument we relien on many properties of the problem (i.e. Brownian motion, one dimension). We would like now to sketch how to solve the conditioning problem for more general processes and more general domains. We will not give all the details or all the necessary assumptions.

Assume $(X_t)_{t\geqslant 0}$ is a Markov diffusion process in $\mathbb{R}^d$ (think to the solution of an SDE with nice coefficients) and let $D\subseteq\mathbb{R}^d$ be a open, connected domain and bounded and let $\tau_D=\inf\{t\geqslant 0: X_t\notin D\}$. Let $\mathbb{P}_x$ be the law of the Markov process starting from $x\in\mathbb{R}^d$ and let $\mathscr{L}$ the generator of the process.



Let us assume that $\mathbb{P}(\tau_D<\infty)=1$. If we want to force the process to stay in D forever we are asking something which has probability 0 under $\mathbb{P}$. So we start by approximating the event we want and just ask that the process stay in D up to time $T>0$ and define $\mathbb{Q}^T$ as the corresponding conditional probability:

$$\mathbb{Q}^T(A):=\frac{\mathbb{E}_{x_0}[\mathbb{1}_A\mathbb{1}_{\tau_D>T}]}{\mathbb{E}[\mathbb{1}_{\tau_D>T}]}.$$

By reasoning as in the previous example we obtain the formula: for any $A\in\mathscr{F}_s$ and $s<T$

$$\mathbb{Q}^T(A)=\frac{\mathbb{E}_{x_0}[\mathbb{1}_A g^{T-s}(X_{s\wedge\tau_D})]}{g^T(x_0)}=\mathbb{E}_{x_0}[\mathbb{1}_A Z_s^T]$$

where

$$g^T(x):=\mathbb{E}_x[\mathbb{1}_{\tau_D>T}].$$

The formula follows by a simple application of the Markov property. It also holds that $Z_s^T:=g^{T-s}(X_{s\wedge\tau_D})/g^T(x_0)$ is a martingale up to time T and $Z_0^T=1$. Note that we have $\mathbb{Q}^T(\tau_D\leqslant T)=0$ so the process X will never hit the boundary before T under $\mathbb{Q}^T$. We will assume that $(T,x)\mapsto g^T(x)$ is $C^{1,2}$, under this condition we can perform Doob's transformation and deduce that if under $\mathbb{P}$ the process X satisfies the SDE

$$\mathrm{d}X_t=b(t,X_t)\mathrm{d}t+\sigma(t,X_t)\mathrm{d}W_t$$

then under $\mathbb{Q}^T$ it will satisfy the SDE (we denote here $\sigma(t,X_t)^*$ the transpose of $\sigma(t,X_t)$)

$$\mathrm{d}X_t=b(t,X_t)\mathrm{d}t+\sigma(t,X_t)\sigma(t,X_t)^*\frac{\nabla g^{T-t}(X_t)}{g^{T-t}(X_t)}\mathrm{d}t+\sigma(t,X_t)\mathrm{d}W_t^T,\qquad t\in[0,T]$$

provided $g^t(x)>0$ for all $t>0$ and $x\in D$. Note that $g^T(x)=0$ if $x\in\partial D$. To take $T\to\infty$ in this SDE is now a bit more difficult that in the BM case because the drift depends on T (and therefore the Brownian motion W^T depends on T). So assumptions have to be made and essentially the most important is to require that the existence of the limit

$$\lim_{T\to\infty}\frac{\nabla g^T(x)}{g^T(x)}$$

and moreover that it is given by

$$\lim_{T\to\infty}\frac{\nabla g^T(x)}{g^T(x)}=\frac{\nabla\varphi_0(x)}{\varphi_0(x)}$$

where φ_0 is the eigenfunction of $(-\mathscr{L})$ with Dirichlet boundary conditions on D and with lowest eigenvalue $\lambda_0 > 0$. The idea is that the function $g^T(x)$ for $T \to \infty$ has the asymptotic expansion

$$g^T(x) = e^{-\lambda_0 T} \varphi_0(x) + o(e^{-\lambda_0 T})$$

uniformly in $x \in D$ and the same for the derivative $\nabla g^T(x)$. Moreover note that the function g if it is $C^{1,2}(\mathbb{R}_+ \times D) \cap C(\bar{D})$ then it is a solution to the parabolic PDE

$$\partial_t g^t(x) = \mathscr{L} g^t(x), \qquad x \in D,$$

$$g^t(x) = 0, \quad x \in \partial D.$$

In order to be sure that these conditions are met one has to make more precise assumptions on b, σ and on D.

$$-\mathscr{L} \varphi_0 = \lambda_0 \varphi_0$$

and $\varphi_0(x) = 0$ for $x \in \partial D$ and $\varphi_0(x) > 0$ for $x \in D$.

In this setting one can prove that the family $(\mathbb{Q}^T)_{T>0}$ weakly convergence to a measure $\mathbb{Q}$ such that the process X is a weak solution to the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\sigma(t, X_t)^* \frac{\nabla \varphi_0(X_t)}{\varphi_0(X_t)} \mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \qquad t \in [0, T].$$

So in general one can expect that a diffusion conditioned to stay inside a given domain satisfy this SDE as soon as we can solve the Dirichlet problem and the solution is suitably regular.

For details see the work of Pisky on Annals of Probability ('80).

Remark 2. This approach cannot be directly used for conditioning BM to stay positive because in this case the function $g^T(x)$ does not decay exponentially in T as $T \to \infty$. The Brownian motion can stay away from zero by going very far out, and this happens with algebraically decaying probability.

Example 3. Take $(X_t)_{t \geqslant 0}$ to be Brownian motion in $d = 1$ and $D = (0, L)$. In this case we can make even precise the above discussion. However the conclusion is that if we condition the BM to stay in D forever it will statisfy the SDE

$$\mathrm{d}X_t = \frac{\nabla \varphi_0(X_t)}{\varphi_0(X_t)} \mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0$$

where φ_0 is the lowest eigenfunction of $-\Delta$ with zero b.c. on $[0, L]$, namely $\varphi_0 = \sin(\pi x / L)$. The eigenvalue $\lambda_0 = (\pi / L)^2$ describe the exponential decay of the probability $\mathbb{P}_x(\tau_D > T) \approx e^{-\lambda_0 T} \varphi_0(x)$. Therefore we have

$$\mathrm{d}X_t = \frac{\pi}{L} \frac{\cos(\pi X_t / L)}{\sin(\pi X_t / L)} \mathrm{d}t + \mathrm{d}W_t^L, \qquad t \geqslant 0.$$

Now if we take (formally) the limit $L \to \infty$ we have

$$\mathrm{d}X_t = \frac{1}{X_t} \mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0.$$

as we expect from our previous computations for the BM conditioned to stay positive.

Example 4. (Brownian motion in the Weyl chamber) Let $X = (X^1, \ldots, X^n)$ be a family of n independent one dimensional BMs and let $S = \{x \in \mathbb{R}^d : x^1 < x^2 < \cdots < x^n\}$ (Weyl chamber) and take $x_0 \in S$ and $X_0 = x_0$. I want to condition X to stay inside S. Consider the function

$$h(x) = \frac{\prod_{i<j} (x^j - x^i)}{\prod_{i<j} (x_0^j - x_0^i)}, \qquad x \in S$$

which is strictly positive in S and 0 on the boundary of S. One can check that this function is harmonic in S, i.e. $(-\Delta_{\mathbb{R}^n})h(x) = 0$ where $\Delta_{\mathbb{R}^n}$ is the Laplacian in $\mathbb{R}^n$ which is the generator of X. Assuming that this is the relevant eigenfunction for describing the conditioning, we get that X conditioned to stay in S solves the SDE

$$\mathrm{d}X^i = \sum_{j \neq i} \frac{1}{X_t^j - X_t^i} \mathrm{d}t + \mathrm{d}W_t^i, \qquad t \geqslant 0.$$