Lecture 13 – 2020.06.02 – 12:15 via Zoom

The student council is organising an event, where a former master student gives an informative talk about her master thesis in the field of Analysis. She just finished her Master's, after having written her thesis in Analysis with Prof. Disertori for one year. The talk will be held next Monday, 08.06. at 18:15. The Zoom link can be found here:

https://fsmath.uni-bonn.de/veranstaltungen-detail/events/mastervorstellung-analysis.html

This talk is designed to give you an idea of what a master thesis can look like and how the process of writing it works.

(from Fachschaft Mathematik)

Some comments about Exercise 2 in Sheet 5. Brownian bridge: BM conditioned to reach a point at a given time. E.g. $y \in \mathbb{R}$ at time $t = 1$. Two ways to construct it. Let $(X_t)_t$ be a Brownian motion in $d = 1$

a) Gaussian approach: define the process

$$X_t^y := X_t + t(X_1 - y),$$

and show that X_1, X^y are independent (via computation of the covariance). So the law μ^y of X^y is the law of the BM conditioned to arrive in y at time 1.

$$\mu^y(\mathrm{d}\omega) := \mathbb{P}(X \in \mathrm{d}\omega | X_t = 1)$$

(we use the fact that BM is a Gaussian process). This method can be used for other Gaussian processes.

b) SDE/Markovian approach (following the method in the lecture) (we use that BM is a Markov process with known transition density, the method works for a large class of Markov processes). The canonical process Y under the conditional law μ^y satisfies the SDE

$$\mathrm{d}Y_t = -\mathbb{1}_{t<1} \frac{Y_t - y}{1-t} \mathrm{d}t + \mathrm{d}B_t$$

The two descriptions have to agree. Indeed note that Y is a Gaussian process (since the SDE is linear): think how to prove it and then check that covariance and mean agree. But of course the construction itself shows that the law of Y and the law of X^y agree.

Note that $(X_t^y)_{t\in[0,1]}$ is not adapted to the filtratrion $\mathscr{F}^X$ of $(X_t)_{t\geqslant 0}$ but it is adapted to the "enlarged filtration" $\mathscr{H}_t := \mathscr{F}_t^X \vee \sigma(X_1)$.

If we consider the process $(X_t^y)_{t\in[0,1]}$ with respect to its own filtration $(\mathscr{G}_t)_{t\geqslant 0}$ (which is smaller than $(\mathscr{H}_t)_t$) then by considering the associated martingale problem and using point b) we deduce that there should exists a Brownian motion $\tilde{B}$ on the same probability space (since $\sigma(x) > 0$ and we are in one dimension, so our simplified argument given at the beginning of the course works) such that

$$X_t^y = -\mathbb{1}_{t<1} \frac{X_t^y - y}{1-t} \mathrm{d}t + \mathrm{d}\tilde{B}_t,$$

This means that $(X_t^y)_t$ is a semimartingale and if we can determine the drift of X_t^y (by some statistical procedure) we can infer the point y (if we know that it is a BM conditioned to arrive somewhere at time 1). At the same time the BM $(\tilde{B})$ contains the "new information" which cannot be predicted by observing the process (and so it is martingale).

So enlarging the filtration by adding the observation $\sigma(X_1)$ to all the σ-fields transforms the BM into the Brownian Bridge above, however it still remains a semimartingale.

There is a whole subfield of stochastic analysis which try to understand how stochastic processes change when enlarging the filtrations (problem of enlargement of filtrations).

SDE techniques: General change of drift

We want now use Girsanov transform to create new processes starting from known ones in more general ways than Doob's transform allows.

For any continuous local martingale $(L_t)_{t\geqslant 0}$ and a probability measure $\mathbb{P}$ on a filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0})$ we can consider the new measure

$$\mathrm{d}\mathbb{Q}^L = \mathscr{E}(L)_\infty \mathrm{d}\mathbb{P}$$

provided $\mathscr{E}(L)_\infty \in L^1(\mathbb{P})$. In this case one can show that $(\mathscr{E}(L)_t)_{t\geqslant 0}$ is a martingale and the Girsanov theorem tells us that

$$\tilde{M}_t = M_t - [L, M]_t, \qquad t \geqslant 0$$

is a continuous local $\mathbb{Q}$-martingale for any local $\mathbb{P}$-martingale M.

So a key point here is how we check that

$$\mathbb{E}[\mathscr{E}(L)_\infty] = \mathbb{E}\left[\exp\left(L_\infty - \frac{1}{2}[L]_\infty\right)\right] = 1,$$

in practice indeed is not obvious how to estimate the local martingale L_∞ appearing there.

Example 1. To keep in mind a useful example one could think that $(X_t)_{t\geqslant 0}$ is a $\mathbb{P}$-BM in $\mathbb{R}^n$ and that we are given a vector field $b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ measurable and let

$$L_t := \int_0^{t\wedge T} b(s, X_s)\mathrm{d}X_s$$

for some T which can be finite or not and maybe a stopping time. In this case we have

$$\mathbb{E}[\mathscr{E}(L)_\infty] = \mathbb{E}\left[\exp\left(\int_0^T b(s, X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^T |b(s, X_s)|^2 \mathrm{d}s\right)\right].$$

In order to estimate this kind of quantities and establish that $(\mathscr{E}(L)_t)_{t\geqslant 0}$ is a martingale it is useful to consider Novikov's condition.

Observe that $(\mathscr{E}(L)_t)_{t\geqslant 0}$ is always a positive local-martingale and is supermartingale (by a Fatou argument) so the point is to show that $\mathbb{E}[\mathscr{E}(L)_\infty] \geqslant 1$ since we know that $\mathbb{E}[\mathscr{E}(L)_\infty] \leqslant 1$.

If you recall the example of the BM with drift in that case one would indeed have $\mathscr{E}(L)_\infty = 0$ a.s.

Novikov condition (in the form given to it by Krylov) is a sufficient criterion to ensure that $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$.

Theorem 2. *(Novikov-Krylov's condition) Let L be a local martingale starting at* 0 *and assume that*

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{E}\left[\exp\left(\frac{1-\varepsilon}{2}[L]_\infty\right)\right] = 0$$

then $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$. *In particular this holds if (this is usually called Novikov's condition)*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}[L]_\infty\right)\right] < \infty.$$

Remark 3. This is not a necessary conditions, it does not care of the sign of L but there are examples where $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$ but $\mathbb{E}[\mathscr{E}(-L)_\infty] < 1$. A finer condition is Kazamaki condition which reads

$$\mathbb{E}[\exp(L_\infty/2)] < \infty$$

but this is not easy to check usually since there are not many ways to estimate the exponential of a stochastic integral.

Example 4. Continuing Example 1. Novikov's condition reads

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |b(s,X_s)|^2 \mathrm{d}s\right)\right] < \infty.$$

It is not difficult to show by using the Markov property of the Brownian motion and the assumption that b is of **linear growth**, i.e. that there exists a constant $C < \infty$ such that

$$|b(t,x)|^2 \leqslant C(1+|x|^2), \qquad x \in \mathbb{R}^n, t \geqslant 0,$$

that Novikov's condition is in this case satisfied. Take $T \geqslant 0$ to be a deterministic time. Then we can defined the measure $\mathbb{Q}^T$ as above and under $\mathbb{Q}^T$ the process

$$\tilde{X}_t = X_t - \int_0^{t \wedge T} b(X_s)\mathrm{d}s, \qquad t \geqslant 0$$

is a $\mathbb{Q}^T$ Brownian motion. (pay attention that this equation is for $\mathbb{R}^n$-valued processes). Now the family of measures $(\mathbb{Q}^T, \mathscr{F}_T)_{t \geqslant 0}$ is a consistent family and therefore it admits a unique extension $\mathbb{Q}^\infty$ to $\mathscr{F}_\infty = \vee_T \mathscr{F}_T$. This happens because the process $(\mathscr{E}(L)_t)_{t \geqslant 0}$ with

$$L_t = \int_0^t b(s,X_s)\mathrm{d}X_s$$

is a martingale for $t \geqslant 0$ excluding $t = \infty$ (is not uniformly integrable in general). Under $\mathbb{Q}^\infty$ we have that X satisfy the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t \qquad t \geqslant 0.$$

for some $\mathbb{Q}^\infty$–Brownian motion B.

Theorem 5. *If b is of linear growth uniformly in time then there exists a weak solution to the SDE in $\mathbb{R}^n$*

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t \qquad t \geqslant 0.$$

We can apply the same method to more general equations. Starting from the solution of an SDE of the form

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t \qquad t \geqslant 0$$

and performing the change of drift to a new measure $\mathbb{Q}$ given by the density $(\mathscr{E}(L)_t)_{t \geqslant 0}$ wrt. $\mathbb{P}$ with

$$L_t = \int_0^t c(X_s)\mathrm{d}B_s$$

(assuming $(\mathscr{E}(L)_t)_{t \geqslant 0}$ is a martingale) we obtain that under $\mathbb{Q}$ the process X is the solution of the SDE ($\tilde{B}_t = B_t - \int_0^t c(X_s)\mathrm{d}s$)

$$\mathrm{d}X_t = (b(X_t) + \sigma(X_t)c(X_t))\mathrm{d}t + \sigma(X_t)\mathrm{d}\tilde{B}_t \qquad t \geqslant 0.$$

So we can change the drift only in directions belonging to the image of $\sigma(x)$.

This observation has application in coupling of diffusions.

Exercise 1. Think about how to perform this change of drift in a martingale problem formulation. (the difficulty is that the is no B in view in the martingale problem). Here is meant without going thru the SDE formulation of martingale problem.

Proof. (of Novikov's condition, via Krylov's proof) The goal is to prove that $\mathbb{E}[\mathscr{E}(L)_\infty]=1$ and by Fatou and a stopping time argument it is enough to check that $\mathbb{E}[\mathscr{E}(L)_\infty]\geqslant 1$.

We start by observing that by Hölder's inequality

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty]=\mathbb{E}\left(\exp\left[(1-\varepsilon)\left(L_\infty-\frac{1}{2}[L]_\infty\right)\right]\exp\left[\frac{\varepsilon(1-\varepsilon)}{2}[L]_\infty\right]\right)$$

$$\leqslant\left\{\mathbb{E}\left(\exp\left[p(1-\varepsilon)\left(L_\infty-\frac{1}{2}[L]_\infty\right)\right]\right)\right\}^{1/p}\left\{\mathbb{E}\left(\exp\left[q\frac{\varepsilon(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^{1/q}$$

$$\leqslant\left\{\mathbb{E}\left(\exp\left[L_\infty-\frac{1}{2}[L]_\infty\right]\right)\right\}^{(1-\varepsilon)}\left\{\mathbb{E}\left(\exp\left[\frac{(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^{\varepsilon}$$

$$\leqslant\{\mathbb{E}(\mathscr{E}(L)_\infty)\}^{(1-\varepsilon)}\left\{\mathbb{E}\left(\exp\left[\frac{(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^{\varepsilon}$$

Now we are in good shape because we just need to control L multiplied with a coefficient less than 1. Therefore is enough to show that $\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty]=1$ given that we know by assumption that

$$\lim_{\varepsilon\downarrow 0}\left\{\mathbb{E}\left(\exp\left[\frac{(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^{\varepsilon}=1.$$

Now we use again the Hölder trick and try to prove that $\mathscr{E}((1-\varepsilon)L)_\infty\in L^p$ for some $p>1$ because in this case I can prove (by localization) that $(\mathscr{E}((1-\varepsilon)L)_t)_{t\geqslant 0}$ is a uniformly integrable martingale and therefore that $\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty]=1$. So take some $p>1$ and observe that (to be done rigorously via a localizing sequence L^{T_n})

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty^p]=\mathbb{E}\left[\exp\left(p(1-\varepsilon)L_\infty-\frac{1}{2}p(1-\varepsilon)^2[L]_\infty\right)\right]$$

one can now apply Hölder to split again this expectation into the form

$$\leqslant\left(\mathbb{E}\left[\exp\left(p'p(1-\varepsilon)\left(L_\infty-\frac{1}{2}[L]_\infty\right)\right)\right]\right)^{1/p'}\left(\mathbb{E}\left[\exp\left(q'\frac{c(p,\varepsilon)}{2}[L]_\infty\right)\right]\right)^{1/q'}$$

take p' so that $p'p(1-\varepsilon)=1$ and observe that (thinking about localization we have $\mathbb{E}[\mathscr{E}(L)_\infty]=1$)

$$\leqslant(\mathbb{E}[\mathscr{E}(L)_\infty])^{1/p'}\left(\mathbb{E}\left[\exp\left(q'\frac{c(p,\varepsilon)}{2}[L]_\infty\right)\right]\right)^{1/q'}$$

$$\leqslant\left(\mathbb{E}\left[\exp\left(q'\frac{c(p,\varepsilon)}{2}[L]_\infty\right)\right]\right)^{1/q'}$$

and now one check that all the coefficients are such there exists an $\varepsilon'\in(0,1)$ (for suitable choice of p and sufficiently small $\varepsilon>0$) such that

$$q'\frac{c(p,\varepsilon)}{2}\leqslant\frac{1-\varepsilon'}{2}$$

which implies

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty^p]\leqslant\left(\mathbb{E}\left[\exp\left(\frac{1-\varepsilon'}{2}[L]_\infty\right)\right]\right)^{1/q'}.$$

Using again our assumption we know that $\mathbb{E}\left[\exp\left(\frac{1-\varepsilon'}{2}[L]_\infty\right)\right]<\infty$ for all $\varepsilon'>0$ so this allow to conclude that

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty^p]<\infty$$

and this finish the proof the theorem. □

Next lecture: uniqueness in law via Girsanov's theorem and maybe the Brownian martingale representation theorem.
