Lecture 14 – 2020.06.04 – 12:15 via Zoom

The student council is organising an event, where a former master student gives an informative talk about her master thesis in the field of Analysis. She just finished her Master's, after having written her thesis in Analysis with Prof. Disertori for one year. The talk will be held next Monday, 08.06. at 18:15. The Zoom link can be found here:

https://fsmath.uni-bonn.de/veranstaltungen-detail/events/mastervorstellung-analysis.html

This talk is designed to give you an idea of what a master thesis can look like and how the process of writing it works.

(from Fachschaft Mathematik)

Uniqueness in law via Girsanov's theorem

Consider the SDE in $\mathbb{R}^n$ with initial condition $X_0 = x_0 \in \mathbb{R}^n$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}B_t, \qquad t \geqslant 0$$

where $b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a measurable time-dependent vector field. We are going to assume that

$$\int_0^T |b(s, X_s)|^2 \mathrm{d}s < +\infty, \qquad \text{a.s. for all } T \geqslant 0. \tag{1}$$

The goal is to show that under this condition all weak solutions of the SDE have the same law, in other words we want to establish uniqueness in law.

We are going to use Girsanov's transformation to remove the drift by absorbing it into the Brownian motion B.

Assume therefore to be given a weak solution (X, B). Define the increasing sequence of stopping times

$$\tau_n := \inf\left\{ t \geqslant 0 : \int_0^t |b(s, X_s)|^2 \mathrm{d}s \geqslant n \right\}.$$

By assumption we have that $\tau_n \to \infty$ a.s. when $n \to \infty$ by (1). Then we can define a new measure $\mathbb{Q}^n$

$$\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^{\tau_n} b(s, X_s)\mathrm{d}B_s - \frac{1}{2}\int_0^{\tau_n} |b(s, X_s)|^2 \mathrm{d}s \right)$$

so that the process

$$\tilde{B}_t = B_t - [L, B]_t = B_t + \int_0^{\tau_n \wedge t} b(s, X_s)\mathrm{d}s$$

is a $\mathbb{Q}^n$ Brownian motion. In particular up to the random time τ_n we have $X_t = \tilde{B}_t$. So $(X_t)_{t \in [0, \tau_n]}$ is a Brownian motion (in the sense that the stopped process X^{τ_n} has the law of a Brownian motion stopped at a stopping time).

Let now $A_T \in \mathscr{B}(\mathscr{C}^n \times \mathscr{C}^n)$ such that $\{(X, B) \in A_T\} \in \mathscr{F}_T$ then

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{(X,B) \in A_T} \mathbb{1}_{T \leqslant \tau_n}] &= \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,B) \in A_T} \mathbb{1}_{T \leqslant \tau_n} \exp\left(\int_0^{\tau_n} b(s, X_s)\mathrm{d}B_s + \frac{1}{2}\int_0^{\tau_n} |b(s, X_s)|^2 \mathrm{d}s \right) \right] \\
&= \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,B) \in A_T} \mathbb{1}_{T \leqslant \tau_n} \exp\left(\int_0^{\tau_n} b(s, X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^{\tau_n} |b(s, X_s)|^2 \mathrm{d}s \right) \right]
\end{aligned}$$

Moreover note that B is an adapted function of X (by the SDE) so $B=\Phi(X)$ where $\Phi\colon \mathscr{C}^n \to \mathscr{C}^n$ is some measurable and adapted functional (recall that $\mathscr{C}^n = C(\mathbb{R}_+;\mathbb{R}^n)$). We write also $\tau_n = \tilde{\tau}_n(X)$ to stress that it is a given measurable function $\tilde{\tau}_n\colon \mathscr{C}^n \to \mathbb{R}_+ \cup \{+\infty\}$ of X. Therefore

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{(X,B)\in A_T}\mathbb{1}_{T\leqslant\tau_n}] &= \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,\Phi(X))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(X)}\exp\left(\int_0^{\tilde{\tau}_n(X)} b(s,X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^{\tilde{\tau}_n(X)} |b(s,X_s)|^2\mathrm{d}s\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,\Phi(X))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(X)}\mathbb{E}_{\mathbb{Q}^n}\left[\exp\left(\int_0^{\tilde{\tau}_n(X)} b(s,X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^{\tilde{\tau}_n(X)} |b(s,X_s)|^2\mathrm{d}s\right)|\mathscr{F}_T\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,\Phi(X))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(X)}\exp\left(\int_0^{\tilde{\tau}_n(X)\wedge T} b(s,X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^{\tilde{\tau}_n(X)\wedge T} |b(s,X_s)|^2\mathrm{d}s\right)\right] \\
&= \int_{\mathscr{C}^n}\mathbb{1}_{(\omega,\Phi(\omega))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(\omega)}\exp\left(\int_0^T b(s,\omega_s)\mathrm{d}\omega_s - \frac{1}{2}\int_0^T |b(s,\omega_s)|^2\mathrm{d}s\right)\mathbb{W}(\mathrm{d}\omega),
\end{aligned}$$

where $\mathbb{W}$ is the law of a $\mathbb{R}^n$ valued Brownian motion (i.e. the Wiener measure). So we proved that the probability $\mathbb{P}((X,B)\in A_T, T\leqslant\tau_n)$ can be expressed independently of the given weak solution and therefore if $(X^1,B^1,\mathbb{P}^1)$ and $(X^2,B^2,\mathbb{P}^2)$ are two weak solutions of the SDE then

$$\mathbb{P}^1((X^1,B^1)\in A_T, T\leqslant\tilde{\tau}_n(X^1)) = \mathbb{P}^2((X^2,B^2)\in A_T, T\leqslant\tilde{\tau}_n(X^2)) \tag{2}$$

moreover if both these weak solutions satisfy the assumptions on the drift we have that

$$\mathbb{P}^1\left(\lim_n \tilde{\tau}_n(X^1)=\infty\right) = \mathbb{P}^2\left(\lim_n \tilde{\tau}_n(X^2)=\infty\right) = 1$$

we can take the limit $n\to\infty$ in (2) and conclude that for any $T\geqslant 0$ and A_T given as above we have

$$\mathbb{P}^1((X^1,B^1)\in A_T) = \mathbb{P}^2((X^2,B^2)\in A_T)$$

which implies uniqueness in law since we can also take $T\to\infty$ to have that

$$\mathbb{P}^1((X^1,B^1)\in A) = \mathbb{P}^2((X^2,B^2)\in A)$$

for any $A\in\mathscr{B}(\mathscr{C}^n\times\mathscr{C}^n)$.

So we proved that

Theorem 1. *The SDE in* $\mathbb{R}^n$

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \mathrm{d}B_t, \qquad t\geqslant 0$$

where $b\colon \mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ *is a measurable time-dependent vector field has uniquess in law in the class of weak solutions which satisfy*

$$\int_0^T |b(s,X_s)|^2\mathrm{d}s < +\infty, \qquad \textit{a.s. for all } T\geqslant 0. \tag{3}$$

In particular, if b is bounded then we have (unconditional) uniqueness in law for the SDE.

Exercise 1. Prove that under the same conditions the unique weak solution X is a Markov process.

Remark 2. The proof works also if $b\colon \mathbb{R}_+\times\mathscr{C}^n\to\mathbb{R}^n$ such that $(b(t,X_t))_{t\geqslant 0}$ is adapted to the filtration generated by X. In this more general context the solution of the SDE it not a Markov process anymore.

Remark that from the proof we have the representation formula

$$\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{(X,B)\in A_T}\mathbb{1}_{T\leqslant\tau_n}] = \int_{\mathscr{C}^n}\mathbb{1}_{(\omega,\Phi(\omega))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(\omega)}\exp\left(\int_0^T b(s,\omega_s)\mathrm{d}\omega_s - \frac{1}{2}\int_0^T |b(s,\omega_s)|^2\mathrm{d}s\right)\mathbb{W}(\mathrm{d}\omega)$$

If we assume that the exponential term is integrable, then we can take by dominated convergence the limit $n \to \infty$ and obtain that

$$\mathbb{P}((X,B) \in A_T) = \int_{\mathscr{C}^n} \mathbb{1}_{(\omega,\Phi(\omega)) \in A_T} \exp\left(\int_0^T b(s,\omega_s) \mathrm{d}\omega_s - \frac{1}{2} \int_0^T |b(s,\omega_s)|^2 \mathrm{d}s \right) \mathbb{W}(\mathrm{d}\omega).$$

In particular one has the explicit representation formula **(path integral formula)**

$$\mathbb{P}(X \in A_T) = \int_{\mathscr{C}^n} \mathbb{1}_{\omega \in A_T} \exp\left(\int_0^T \langle b(s,\omega_s), \mathrm{d}\omega_s \rangle_{\mathbb{R}^n} - \frac{1}{2} \int_0^T |b(s,\omega_s)|_{\mathbb{R}^n}^2 \mathrm{d}s \right) \mathbb{W}(\mathrm{d}\omega) \tag{4}$$

for any $A_T \in \sigma(\omega_t : t \in [0,T]) \subseteq \mathscr{B}(\mathscr{C}^n)$. For example this would hold if Novikov's condition is satisfied

$$\int_{\mathscr{C}^n} \exp\left(\frac{1}{2} \int_0^T |b(s,\omega_s)|^2 \mathrm{d}s \right) \mathbb{W}(\mathrm{d}\omega) < \infty.$$

It could be tempting to try to use the formula (4) to simulate a diffusion, indeed by Monte-Carlo methods one could take independent samples $(B^{(k)})_{k \in \mathbb{N}}$ of a Brownian motion and observe that by the law of large numbers one has

$$\mathbb{P}(X \in A_T) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{B^{(k)} \in A_T} \exp\left(\int_0^T \langle b(s,B_s^{(k)}), \mathrm{d}B_s^{(k)} \rangle_{\mathbb{R}^n} - \frac{1}{2} \int_0^T |b(s,B_s^{(k)})|_{\mathbb{R}^n}^2 \mathrm{d}s \right)$$

the appeal of this method would be that it is very easy to simulate an (approximate) Brownian motion (i.e. via the Levy construction). Unfortunately is not easy to have a robust approximation of the stochastic integral in the exponent: i.e. if one try to replace it by Riemann sums then the resulting object converge very slowly to its “real value” and moreover it show very wild oscillations due to the fact that the exponential function “amplifies” very large positive fluctuations of its argument (all these problems are “similar” or “of the same nature” of the subtelties related to the integrability of the stochastic exponential $\mathscr{E}(L)$).

A particular situation which is quite nice is when $b(x) = -\nabla V(x)$ with a sufficiently smooth function V. Indeed in this case we have, by Ito formula on the canonical space $\mathscr{C}^n$ with the Wiener measure $\mathbb{W}$:

$$V(\omega_T) = V(\omega_0) + \int_0^T \nabla V(\omega_s) \mathrm{d}\omega_s + \frac{1}{2} \int_0^T \Delta V(\omega_s) \mathrm{d}s$$

provided $V \in C^2(\mathbb{R}^n)$ so that by “integrating by parts” we have

$$\exp\left(\int_0^T \langle b(\omega_s), \mathrm{d}\omega_s \rangle_{\mathbb{R}^n} - \frac{1}{2} \int_0^T |b(\omega_s)|_{\mathbb{R}^n}^2 \mathrm{d}s \right) = \exp\left(-\int_0^T \nabla V(\omega_s) \mathrm{d}\omega_s - \frac{1}{2} \int_0^T |\nabla V(s,\omega_s)|^2 \mathrm{d}s \right)$$

$$= \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2} \int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s)) \mathrm{d}s \right) = \Phi(\omega)$$

and the stochastic integral disappear from the exponent. This make the numerical method more stable since now the functional $\Phi : \mathscr{C}^n \to \mathbb{R}_+$ is easily seen to be continuous in the uniform topology on $\mathscr{C}^n$.

The formula

$$\mathbb{P}(X \in A_T) = \int_{\mathscr{C}^n} \mathbb{1}_{\omega \in A_T} \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2} \int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s)) \mathrm{d}s \right) \mathbb{W}(\mathrm{d}\omega)$$

can be also used to understand other properties of the solutions X of the SDE. Take for example $X_0 = x$ (call $\mathbb{P}_x$ the law of the associated solution to the SDE) and $f : \mathbb{R}^n \to \mathbb{R}$ and observe that

$$\mathbb{E}_x(f(X_T)) = \int_{\mathscr{C}^n} f(\omega_T) \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2} \int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s)) \mathrm{d}s \right) \mathbb{W}_x(\mathrm{d}\omega)$$

where $\mathbb{W}_x$ is the Wiener measure starting from x, i.e. with $\omega_0 = x$ almost surely. So we can express the transition kernel P of the time-homogeneous markov process (X_t) as

$$(P_T f)(x) = \mathbb{E}_x(f(X_T)) = \int_{\mathscr{C}^n} f(\omega_T) \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2}\int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s)) \mathrm{d}s \right) \mathbb{W}_x(\mathrm{d}\omega).$$

$$|(P_T f)(x)| \leqslant \|f e^{-V}\|_\infty e^{V(x)} \exp\left(-\frac{1}{2}\int_0^T \inf_{x\in\mathbb{R}^n} (|\nabla V(x)|^2 - \Delta V(x)) \mathrm{d}s \right).$$

So for example, if $\inf_{x\in\mathbb{R}^n} (|\nabla V(x)|^2 - \Delta V(x)) \geqslant 2\alpha > 0$ then we have the exponential decay

$$e^{-V(x)} |(P_T f)(x)| \leqslant e^{-\alpha T} \|f e^{-V}\|_\infty,$$

in other words

$$\|e^{-V} (P_T f)\|_\infty \leqslant e^{-\alpha T} \|f e^{-V}\|_\infty.$$

Exercise 2. Using the path-integral formula show that for any two bounded functions f, g and under appropriate conditions on V:

$$\int (P_T f)(x) g(x) e^{-V(x)} \mathrm{d}x = \int f(x) (P_T g)(x) e^{-V(x)} \mathrm{d}x$$

which shows that P_T is symmetric wrt. the measure $e^{-V(x)}\mathrm{d}x$ and taking $g = 1$ show that $e^{-V(x)}\mathrm{d}x$ properly normalized is an invariant measure for the SDE

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \mathrm{d}B_t,$$

meaning that if X_0 is taken with probability distribution $\propto e^{-V(x)}\mathrm{d}x$ then

$$\mathbb{E}[f(X_0)] = \mathbb{E}[f(X_T)],$$

for all $T \geqslant 0$.

Remark on Ex 3 of Sheet 6:

Note the relevant Hilbert space is $L^2(\mathbb{R}^n)$ where

$$\int \overline{f(x)} \nabla_\alpha g(x) \mathrm{d}x = \int \overline{(-\nabla_\alpha) f(x)} g(x) \mathrm{d}x$$

so $\nabla_\alpha^* = -\nabla_\alpha$

$$H(A) f = |\nabla - iA|^2 f + Vf = \sum_{\alpha=1}^n (\nabla_\alpha - iA_\alpha)^* (\nabla_\alpha - iA_\alpha) f + Vf$$

$$= \sum_{\alpha=1}^n (-\nabla_\alpha + iA_\alpha)(\nabla_\alpha - iA_\alpha) f = \sum_{\alpha=1}^n (-\nabla_\alpha \nabla_\alpha f + i\nabla_\alpha (A_\alpha f) + iA_\alpha \nabla_\alpha f + A_\alpha^2 f)$$

$$= -\Delta f + i2A \cdot \nabla f + (i(\nabla \cdot A) + |A|^2) f$$

From the rep. formula by using Jensen's inequality and taking $\psi_0 \geqslant 0$

$$|(e^{-H(A)t}\psi_0)(x)| \leqslant (e^{-H(0)t}\psi_0)(x)$$

$$\psi(t,x) = \sum_{n\geqslant 0} e^{-E_n t} \langle \varphi_n, \psi_0 \rangle \varphi_n(x) = e^{-E_0 t} \langle \varphi_0, \psi_0 \rangle \varphi_0(x) + e^{-E_0 t} \sum_n e^{-(E_n - E_0)t} \langle \varphi_n, \psi_0 \rangle \varphi_n(x)$$

Suggestion take ψ_0 to be the lowest eigenfunction of either $H(A)$ or $H(0)$.

Next week: local times and Ito-Tanaka formula.