Lecture 15 – 2020.06.09 – 12:15 via Zoom

Ito–Tanaka formula and local times of semimartingales

We want to extend Ito formula to functions which are not C^2.
Let X be a (one-dimensional) semimartingale and $f:\mathbb{R}\to\mathbb{R}$ a convex function.
Recall that for f convex there always exists f_-' (the derivative from the left) and it is an increasing function.
Let $\rho\in C^\infty(\mathbb{R})$ which is compactly supported on $\{x<0\}$, for example in $(-1,0)$ and define

$$f_n(x):=n\int \rho(ny)f(x+y)\mathrm{d}y$$

which is a smooth function such that $f_n\to f$ pointwise and for which $f_n'(x)\nearrow f_-'(x)$. By Ito formula

$$f_n(X_t)=f_n(X_0)+\int_0^t f_n'(X_s)\mathrm{d}X_s+\frac{1}{2}A_t^{f_n}$$

with $A_t^{f_n}:=\int_0^t f_n''(X_s)\mathrm{d}[X]_s$ a continuous, increasing process. Eventually by using stopping times we can localize the problem so that f_-' is bounded, morover we note that by Doob's inequality we have (where $X=M+V$ is the decomposition of the semimartingale)

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t (f_n'(X_s)-f_-'(X_s))\mathrm{d}M_s\right|^2\right]\lesssim\mathbb{E}\left[\left|\int_0^T (f_n'(X_s)-f_-'(X_s))\mathrm{d}M_s\right|^2\right]$$
$$\lesssim\mathbb{E}\left[\int_0^T (f_n'(X_s)-f_-'(X_s))^2\mathrm{d}[M]_s\right]\to 0$$

by dominated convergence (again maybe put a stopping time to guarantee boundedness). This shows that in probability and uniformly on compact sets (in t)

$$\int_0^t f_n'(X_s)\mathrm{d}M_s\to\int_0^t f_-'(X_s)\mathrm{d}M_s.$$

On the hand, always by dominated convergence (decomposing the finite measure $\mathrm{d}V_s$ into positive and negative parts)

$$\int_0^t f_n'(X_s)\mathrm{d}V_s\to\int_0^t f_-'(X_s)\mathrm{d}V_s.$$

We can conclude that we have the following lemma

Lemma 1. *If X is a continuous semimartingale and f a convex function, then there exists a continuous increasing process $(A_t^f)_t$ such that*

$$f(X_t)=f(X_0)+\int_0^t f_-'(X_s)\mathrm{d}X_s+\frac{1}{2}A_t^f,\qquad t\geqslant 0.$$

We now can take $f(x)$ nice and simple convex functions like $|x-a|$, $(x-a)_\pm$ where $(x)_+=(x\wedge 0)$ and $(x)_-:=(-x)_+$. As a corollary of the previous lemma we then have

Theorem 2. *(Tanaka's formula) For any $a\in\mathbb{R}$ there exists a continuous increasing process $(L_t^a)_{t\geqslant 0}$ such that*

$$|X_t-a|=|X_0-a|+\int_0^t \mathrm{sgn}(X_s-a)\mathrm{d}X_s+L_t^a$$

$$(X_t-a)_+=(X_0-a)_++\int_0^t \mathbb{1}_{X_s>a}\mathrm{d}X_s+\frac{1}{2}L_t^a$$

$$(X_t-a)_-=(X_0-a)_--\int_0^t \mathbb{1}_{X_s\leqslant a}\mathrm{d}X_s+\frac{1}{2}L_t^a$$

where $\operatorname{sgn}(x) = \mathbb{1}_{x>0} - \mathbb{1}_{x\leqslant 0}$.

Remark 3. This proves in particular that $|X_t - a|$, $(X_t - a)_\pm$ are semimartingales. The process $(L_t^a)_{t\geqslant 0}$ it is called the local time of X at a.

Proof. Each of the formulas derives from the previous lemma by computing the left derivative of the various convex functions. The missing point is to identify the various increasing processes $A^{\operatorname{sgn}(x-a)}, A^{(x-a)_+}$, $A^{(x-a)_-}$. Note that

$$X_t - a = (X_t - a)_+ - (X_t - a)_- = X_0 - a + \int_0^t \underbrace{(\mathbb{1}_{X_s>a} + \mathbb{1}_{X_s\leqslant a})}_{=1} \mathrm{d}X_s + \frac{1}{2}(A_t^{(x-a)_+} - A_t^{(x-a)_-})$$

so we have

$$0 = X_t - X_0 - \int_0^t \mathrm{d}X_s = \frac{1}{2}(A_t^{(x-a)_+} - A_t^{(x-a)_-}) \Rightarrow A_t^{(x-a)_+} = A_t^{(x-a)_-} =: L_t^a.$$

Moreover

$$|X_t - a| = (X_t - a)_+ + (X_t - a)_- + \int_0^t \underbrace{(\mathbb{1}_{X_s>a} - \mathbb{1}_{X_s\leqslant a})}_{=\operatorname{sgn}(X_s - a)} \mathrm{d}X_s + \underbrace{\frac{1}{2}(A_t^{(x-a)_+} + A_t^{(x-a)_-})}_{L_t^a}$$

□

The increasing process $(L_t^a)_{t\geqslant 0}$ is associated with a measure $\mathrm{d}L_t^a$ on $\mathbb{R}_+$ (times) which represents the time the process X "spent" in a up to time t. We are going to make this precise in the following.

By Ito formula wrt. the semimartingale $(|X_t - a|)_{t\geqslant 0}$ (with $X = M + V$)

$$(X_t - a)^2 = (|X_t - a|)^2 = (|X_0 - a|)^2 + 2\int_0^t |X_s - a| \operatorname{sgn}(X_s - a)\mathrm{d}X_s + 2\int_0^t |X_s - a| \mathrm{d}L_s^a + [|X_\cdot - a|]_t$$

$$= (X_0 - a)^2 + 2\int_0^t (X_s - a)\mathrm{d}X_s + 2\int_0^t |X_s - a|\mathrm{d}L_s^a + \int_0^t \underbrace{\operatorname{sgn}(X_s - a)^2}_{=1} \mathrm{d}[M]_s$$

And by comparing with the standard Ito formula

$$(X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a)\mathrm{d}X_s + \underbrace{[X]_t}_{=[M]_t}$$

we conclude that

$$\int_0^t |X_s - a|\mathrm{d}L_s^a = 0, \qquad t \geqslant 0$$

which proves that the measure $(\mathrm{d}L_s^a)_{s\geqslant 0}$ is supported in the (random) set $\{s \in \mathbb{R} : X_s = a\}$ of times. The process L^a increases only when the process X visits a (in general this will be a "fractal-like" and with zero Lebesgue measure).

For Brownian motion is it true (we will not prove it) that the set $\{s \in \mathbb{R} : X_s = a\}$ is the support of the measure $(L_t^a)_{t\geqslant 0}$.

Theorem 4. *(Ito–Tanaka formula) If f is the difference of two convex functions and X a continuous semimartingale, then*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(\mathrm{d}a)$$

and in particular $(f(X_t))_{t\geqslant 0}$ is a semimartingale.

In this formula $f''(\mathrm{d}a)$ denotes the measure associated to the second derivative of a convex function (and therefore of a difference of two convex functions, by linearity).

Recall that for any convex function f we have the formula

$$f(x) = \alpha + \beta x + \frac{1}{2}\int |x-a| f''(\mathrm{d}a), \qquad x \in \mathbb{R}$$

and

$$f_-'(x) = \beta + \frac{1}{2}\int \operatorname{sgn}(x-a) f''(\mathrm{d}a), \qquad x \in \mathbb{R}$$

for some $\alpha, \beta \in \mathbb{R}$ and $f''(\mathrm{d}a)$ is the measure associated to the increasing function $(f_-'(x))_{x\in\mathbb{R}}$ (convex functions are those functions whose second distributional derivative is a positive Radon measure).

The idea is that

$$\frac{\mathrm{d}}{\mathrm{d}x}|x-a| = 2\delta(x-a)$$

which justify intuitively the $1/2$ in the formula.

Proof. We can write

$$f(X_t) = \alpha + \beta X_t + \frac{1}{2}\int |X_t - a| f''(\mathrm{d}a)$$

by Tanaka's formula

$$f(X_t) = \alpha + \beta X_0 + \beta \int_0 \mathrm{d}X_s + \frac{1}{2}\int |X_0 - a| f''(\mathrm{d}a) + \frac{1}{2}\int \left(\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s\right) f''(\mathrm{d}a)$$
$$+ \frac{1}{2}\int L_t^a f''(\mathrm{d}a)$$

Note that this computation makes sense since the stochastic integral $\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s$ is a measurable function of a, more precisely (see the relevant exercise in Sheet 7) the function

$$(a,t,\omega) \mapsto \left(\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s\right)(\omega)$$

is a measurable function on $\mathscr{B}(\mathbb{R}) \otimes \mathscr{P}$ ($\mathscr{P}$ is the previsible σ field on $\mathbb{R}_+ \times \Omega$) and also a (stochastic) Fubini theorem applies so that

$$\int \left(\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s\right) f''(\mathrm{d}a) = \int_0^t \left(\int \operatorname{sgn}(X_s - a) f''(\mathrm{d}a)\right)\mathrm{d}X_s$$

Moreover we note that

$$\beta \int_0^t \mathrm{d}X_s + \frac{1}{2}\int_0^t \left(\int \operatorname{sgn}(X_s - a) f''(\mathrm{d}a)\right)\mathrm{d}X_s = \int_0^t f_-'(X_s)\mathrm{d}X_s$$

which completes the proof. □

Corollary 5. *(Occupation-time formula) There is a $\mathbb{P}$-negligible set $\mathscr{N}$ outside which for any $t \geqslant 0$ and any positive Borel function $g\colon \mathbb{R} \to \mathbb{R}_+$ we have*

$$\int_0^t g(X_s)\mathrm{d}[X]_s = \int_{\mathbb{R}} g(a) L_t^a \mathrm{d}a.$$

Remark 6. The measure $\mathrm{d}[X]_s$ can be understood as some "intrinsic" time of the semimartingale. In particular, for Brownian motion X we have $\mathrm{d}[X]_s = \mathrm{d}s$ and if we take $g(x) = \mathbb{1}_{x\in A}$ for some set $A \in \mathscr{B}(\mathbb{R})$ we have

$$\operatorname{Leb}(\{s \in [0,t]\colon X_s \in A\}) = \int_0^t \mathbb{1}_{X_s \in A}\mathrm{d}s = \int_A L_t^a \mathrm{d}a.$$

In this sense $L_t^a \mathrm{d}a$ represents the time spent by X in the infinitesimal neighborhood $a \pm \mathrm{d}a$.

Proof. For any $g:\mathbb{R}\to\mathbb{R}_+$ we can find a convex function f such that $f''=g$, i.e. we can take $f''(\mathrm{d}a)=g(a)\mathrm{d}a$ in the formula above. By Tanaka's formula we then have

$$f(X_t)=f(X_0)+\int_0^t f'_-(X_s)\mathrm{d}X_s+\frac{1}{2}\int_{\mathbb{R}} g(a)L_t^a\mathrm{d}a. \qquad \mathbb{P}-a.s.$$

Take a countable family $(g_n)_{n\geqslant 0}$ of compactly supported continuous functions which is dense in $C_0(\mathbb{R})$ and consider now f_n so that $f_n''=g_n$, note that $f_n\in C^2$ and $f'_{n,-}=f'_{n,+}=f'_n$. I have now both Ito-Tanaka's formula and Ito formula (note that f_n is the difference of two convex functions)

$$f_n(X_t)=f_n(X_0)+\int_0^t f_n'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^t g_n(X_s)\mathrm{d}[X]_s. \qquad \mathbb{P}-a.s.$$

So by comparing these two formulas we have

$$\ell_t(g_n):=\int_{\mathbb{R}} g_n(a)L_t^a\mathrm{d}a=\int_0^t g_n(X_s)\mathrm{d}[X]_s. \qquad \mathbb{P}-a.s.$$

This equality holds a.s. for any g_n and one can choose a $\mathbb{P}$-negligible set $\mathcal{N}$ such that the equalities holds simultaneously for all n and all $t\geqslant 0$ (since the quantity $\ell_t(g_n)$ is continuous in time and therefore can be detemined by looking to a dense set of times $(t_k)_k$).

One note now that for any $t\geqslant 0$, the functional ℓ_t is a positive linear functional on $C_0(\mathbb{R})$ which is continuous in the uniform norm on $C_0(\mathbb{R})$ so can be extended by continuity to all functions in $C_0(\mathbb{R})$ and by a monotone class argument to all Borel positive functions. □

Thursday no lecture.

Next lecture on tuesday: we prove that $a\mapsto L_t^a$ is cadlag and that there is a formula of the form

$$L_t^a=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{X_s\in(a,a+\varepsilon)}\mathrm{d}[X]_s$$

and for X a martingale

$$L_t^a=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{X_s\in(a-\varepsilon,a+\varepsilon)}\mathrm{d}[X]_s.$$

We continue to discuss some properties of local time of Brownian motion and reflected Brownian motion.

For Exercise Sheet 7, exercise 2, the invariant measure should be e^{-2V} and not e^{-V}.