Lecture 16 – 2020.06.16 – 12:15 via Zoom

Regularity of local times and reflected Brownian motion, Takana's SDE

We want to look at $a \mapsto L_t^a$ where L_t^a is the local time in a of a semimartingale X.

Recall the occupation time formula

$$\int_0^t \varphi(X_t)\mathrm{d}[X]_t = \int_{\mathbb{R}} \varphi(x) L_t^x \mathrm{d}x$$

for all $t \geqslant 0$ and $\varphi: \mathbb{R} \to \mathbb{R}_+$ positive bounded Borel function.

Remark 1. Note that using this formula one can prove that Ito formula extends to any f such that f'' is locally integrable with respect to the Lebesgue measure, i.e.

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} f''(x) L_t^x \mathrm{d}x$$

Observe that

$$f'(x) - f'(y) = \int_x^y f''(z)\mathrm{d}z$$

so f' is of bounded variation and, by dominated convergence, continuous.

So far we know only that $a \mapsto L_t^a$ is measurable in a. Denote $X = M + V$

Tanaka's formula give

$$L_t^a = 2\Big[(X_t - a)_+ - (X_0 - a)_+ - \int_0^t \mathbb{1}_{X_s > a}\mathrm{d}M_s - \int_0^t \mathbb{1}_{X_s > a}\mathrm{d}V_s\Big]$$

Define

$$\hat{M}_t^a := \int_0^t \mathbb{1}_{X_s > a}\mathrm{d}M_s$$

We want to apply Kolmogorov's continuity theorem to $a \in \mathbb{R} \mapsto (\hat{M}_t^a)_{t \in [0,T]} \in C([0,T];\mathbb{R})$ seen as a random variable with values on $C_T = C([0,T];\mathbb{R})$ with norm $\|f\|_{C_T} = \sup_{t \in [0,T]} \|f(t)\|$. Recall that Kolmogorov's continuity theorem states that a stochastic process $Y: \mathbb{R} \to \mathscr{B}$ has a continuous version if

$$\mathbb{E}\|Y(a) - Y(b)\|_{\mathscr{B}}^p \leqslant C_L |a-b|^{1+c}$$

for some $p, c > 0$ and $a, b \in [0, L]$ for all L with some finite C_L. Moreover a consequence of the theorem is alos that the process Y can be chosen to be locally Hölder continuous with index $\gamma \in (0, c/p)$, namely for any $L > 0$

$$\|Y(a)(\omega) - Y(b)(\omega)\|_{\mathscr{B}} \leqslant K_L(\omega)|a-b|^{\gamma}, \qquad a, b \in [0, L]$$

almost surely.

In our case we take $\mathscr{B} = C_T$ and then we need to estimate for some $p \geqslant 2$

$$\mathbb{E}[\sup_{t \in [0,T]} |\hat{M}_t^a - \hat{M}_t^b|^p]$$

By Burkholder-David-Gundy (BDG) inequality, (see next exercise sheet) take $b > a$,

$$\mathbb{E}[\sup_{t \in [0,T]} |\hat{M}_t^a - \hat{M}_t^b|^p] \leqslant C_p \mathbb{E}[[\hat{M}^a - \hat{M}^b]_T^{p/2}]$$

$$\leqslant C_p \mathbb{E}\Big[\Big(\int_0^T (\mathbb{1}_{X_s > a} - \mathbb{1}_{X_s > b})^2 \mathrm{d}[M]_s\Big)^{p/2}\Big]$$

by occupation time formula

$$\lesssim C_p \mathbb{E}\Big[\Big(\int_a^b L_T^x \mathrm{d}x\Big)^{p/2}\Big]$$

by Jensen's inequality

$$\lesssim C_p (b-a)^{p/2} \mathbb{E}\Big[\int_a^b (L_T^x)^{p/2} \frac{\mathrm{d}x}{b-a}\Big] \lesssim_p (b-a)^{p/2} \sup_{x\in\mathbb{R}} \mathbb{E}[(L_T^x)^{p/2}].$$

In order to show that $\sup_{x\in\mathbb{R}} \mathbb{E}[(L_T^x)^{p/2}]$ is finite we observe that since

$$|(X_T-a)_+ - (X_0-a)_+| \leqslant |X_T - X_0|$$

$$\mathbb{E}[(L_T^x)^{p/2}] = \mathbb{E}\Big[\Big(2\Big[(X_T-a)_+ - (X_0-a)_+ - \int_0^T \mathbb{1}_{X_s>a} \mathrm{d}M_s - \int_0^T \mathbb{1}_{X_s>a} \mathrm{d}V_s\Big]\Big)^{p/2}\Big]$$

$$\lesssim_p \mathbb{E}[|X_T - X_0|^{p/2}] + \mathbb{E}\Big[\Big|\int_0^T \mathbb{1}_{X_s>a} \mathrm{d}M_s\Big|^{p/2}\Big] + \mathbb{E}\Big[\Big|\int_0^T \mathbb{1}_{X_s>a} \mathrm{d}V_s\Big|^{p/2}\Big]$$

$$\lesssim_p \mathbb{E}[|X_T - X_0|^{p/2}] + \mathbb{E}[|[M]_T|^{p/4}] + \mathbb{E}\Big[\Big(\int_0^T |\mathrm{d}V_s|\Big)^{p/2}\Big] = L_T$$

this shows that $\sup_{x\in\mathbb{R}} \mathbb{E}[(L_T^x)^{p/2}]$ is finite provided L_T is finite. In this case Kolmogorov's continuity criterion tells us that $a \mapsto \hat{M}_t^a$ is continuous in a uniformly in t. If the quantity L_T is not finite, then we introduce a suitable sequence of stopping times $(T_n)_n$, $T_n \to \infty$ and look at that stopped martingale $(\hat{M}_t^a)_t^{T_n}$. For example take

$$T_n = \inf\Big\{t \geqslant 0 \colon \sup_{s\in[0,t]} |X_s - X_0| + [M]_t + \int_0^t |\mathrm{d}V_s| \geqslant n\Big\}$$

so that we now know that $(t,a) \mapsto (\hat{M}_t^a)^{T_n}$ is continuous in both variables and then taking the limit as $n \to \infty$ we deduce that $(t,a) \mapsto \hat{M}_t^a$ is also continuous in both variables since $T_n \to \infty$ almost surely.

Actually from this proof one could also deduce that the process $a \mapsto \hat{M}_t^a$ for fixed t is locally Hölder continuous for any $\gamma < 1/2$, i.e.

$$\sup_{t\in[0,T]} |\hat{M}_t^a - \hat{M}_t^b| \leqslant C_L(\omega) |b-a|^\gamma, \qquad a,b \in [0,L].$$

holds almost surely for some random constant C_L which can be taken to be

$$C_L(\omega) = C_L^{N_T}(\omega)$$

where $N_T := \inf_{n\geqslant 0}\{n \colon T_n > T\}$ where $C_L^{N_T}(\omega)$ is the constant appearing in the bound

$$\sup_{t\in[0,T_n]} |\hat{M}_t^a - \hat{M}_t^b| \leqslant C_L^n(\omega) |b-a|^\gamma, \qquad a,b \in [0,L]$$

which holds for any $n \geqslant 0$ by considering the stopped process.

As far as $\int_0^t \mathbb{1}_{X_s>a} \mathrm{d}V_s$ is concerned we have letting

$$\hat{V}_t^a := \int_0^t \mathbb{1}_{X_s>a} \mathrm{d}V_s$$

and using dominated convergence

$$\hat{V}_t^{a+} = \lim_{b\searrow a} \hat{V}_t^b = \int_0^t \mathbb{1}_{X_s>a} \mathrm{d}V_s = \hat{V}_t^a$$

since $\lim_{b\searrow a} \mathbb{1}_{X_s>b} = \mathbb{1}_{X_s>a}$. However we have $\lim_{b\nearrow a} \mathbb{1}_{X_s>b} = \mathbb{1}_{X_s\geqslant a}$ so

$$\hat{V}_t^{a-} = \lim_{b\nearrow a} \hat{V}_t^b = \int_0^t \mathbb{1}_{X_s\geqslant a} \mathrm{d}V_s \neq \hat{V}_t^a$$

So the process $a \mapsto \hat{V}_t^a$ is almost surely cadlag. Additionally

$$\hat{V}_t^{a-} - \hat{V}_t^a = \int_0^t \mathbb{1}_{X_s = a} \mathrm{d}V_s = \int_0^t \mathbb{1}_{X_s = a} \mathrm{d}X_s$$

since by the occupation time formula and Ito isometry, we have $\int_0^t \mathbb{1}_{X_s = a} \mathrm{d}M_s = 0$, since

$$\left[\int_0^{\cdot} \mathbb{1}_{X_s = a} \mathrm{d}M_s\right]_T = \int_0^T \mathbb{1}_{X_s = a} \mathrm{d}[M]_s = \int_0^T \mathbb{1}_{X_s = a} \mathrm{d}[X]_s = \int_{\mathbb{R}} \mathbb{1}_{x = a} L_T^x \mathrm{d}x = 0$$

almost surely. Putting all together we have proven the following theorem

Theorem 2. *For any continuous semimartingale X there exists a modification of the local time process $(L_t^a)_{t,a}$ which is continuous in t and cadlag in a and moreover we have*

$$L_t^a - L_t^{a-} = 2\int_0^t \mathbb{1}_{X_s = a} \mathrm{d}X_s = 2\int_0^t \mathbb{1}_{X_s = a} \mathrm{d}V_s.$$

In particular, if X is a local martingale then the local time has bicontinuous version.

Corollary 3. *If X is a continuous semimartingale then*

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{X_s \in [a, a+\varepsilon[} \mathrm{d}[X]_s$$

and if X is a martingale

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{X_s \in]a-\varepsilon, a+\varepsilon[} \mathrm{d}[X]_s$$

Proof. Just use the occupation time formula and the continuity from the right of local times. □

Remark 4. For Brownian motion this implies that L_t^0 is measurable with respect to the filtration $\mathscr{F}^{|B|}$ generated by $|B|$ since $\mathbb{1}_{B_s \in]-\varepsilon, +\varepsilon[} = \mathbb{1}_{|B_s| < \varepsilon} \hat{\in} \mathscr{F}^{|B|}$ and $[B]_s = s$ and

$$L_t^0 = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{|B_s| < \varepsilon} \mathrm{d}s.$$

Brownian motion and local time

Let B be a one dimensional Brownian motion. By Ito–Tanaka formula we have

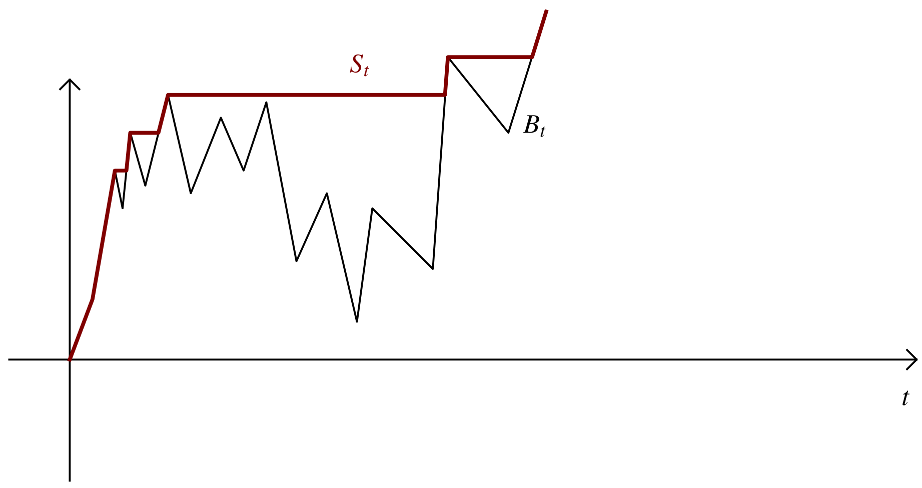
$$|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s) \mathrm{d}B_s + L_t \tag{1}$$

where we let L_t to be the local time in zero of B. Is not important in this case to specify which version of the sign it is used since by the occupation time formula

$$\left[\int_0^{\cdot} \mathbb{1}_{B_s = 0} \mathrm{d}B_s\right]_T = \int_0^T \mathbb{1}_{B_s = 0} \mathrm{d}s = \int_{\mathbb{R}} \mathbb{1}_{x = 0} L_T^x \mathrm{d}x = 0.$$

We want to show next that $R_t = |B_t|$ is an interesting process which satisfies a *reflected* SDE and is called the reflected Brownian motion, this will make link also with another process which is the maximum of the Brownian motion

$$S_t := \sup_{s \leqslant t} B_s$$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