Lecture 17 – 2020.06.18 – 12:15 via Zoom

Brownian motion and local time

Let B be a one dimensional Brownian motion starting in 0. By Ito–Tanaka formula we have

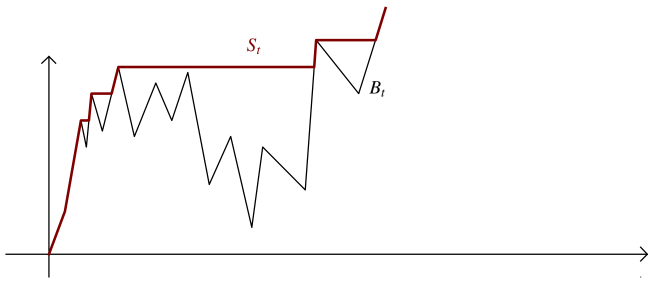
$$|B_t| = \int_0^t \mathrm{sgn}(B_s)\mathrm{d}B_s + L_t \tag{1}$$

where we let L_t to be the local time in zero of B. Is not important in this case to specify which version of the sign it is used since by the occupation time formula

$$\left[\int_0^\cdot \mathbb{1}_{B_s=0}\mathrm{d}B_s\right]_T = \int_0^T \mathbb{1}_{B_s=0}\mathrm{d}s = \int_{\mathbb{R}} \mathbb{1}_{x=0}L_T^x\mathrm{d}x = 0.$$

We want to show next that $R_t = |B_t|$ is an interesting process which satisfies a *reflected* SDE and is called the reflected Brownian motion, this will make link also with another process which is the maximum of the Brownian motion

$$S_t^B := \sup_{s \leqslant t} B_s$$



Take again

$$R_t = |B_t| = \int_0^t \mathrm{sgn}(B_s)\mathrm{d}B_s + L_t$$

and define

$$\beta_t := \int_0^t \mathrm{sgn}_{-1}(B_s)\mathrm{d}B_s$$

where we denote sgn_a the signum function which satisfy $\mathrm{sgn}_a(0) = a$. Note that sgn_{-1} is the left derivative of the absolute value.

Observe by Lévy characterisation that β is a Brownian motion, indeed $[\beta]_t = \int_0^t \mathrm{sgn}_{-1}(B_s)^2\mathrm{d}[B]_s = [B]_t = t$, moreover

$$\int_0^t \mathrm{sgn}_0(B_s)\mathrm{d}\beta_s = \int_0^t \mathrm{sgn}_0(B_s)\mathrm{sgn}_{-1}(B_s)\mathrm{d}B_s = \int_0^t \mathrm{sgn}_0(B_s)^2\mathrm{d}B_s = B_t - \underbrace{\int_0^t \mathbb{1}_{B_s=0}\mathrm{d}B_s}_{=0} = B_t$$

since using the local time of B I have $[\int_0^\cdot \mathbb{1}_{B_s=0}\mathrm{d}B_s]_\infty = 0$.

The first observation out of this computation is that (B, β) is a weak solution of the SDE

$$\mathrm{d}B_t = \mathrm{sgn}_0(B_t)\mathrm{d}\beta_t,$$

this is called *Tanaka's* SDE. So we have proven weak existence for this equation. This solution is unique in law (obviously) since any solution will be such that B is a Brownian motion. However this SDE do not have strong solutions. Indeed if (X, W) is a strong solution (starting in $X_0 = 0$), we have

$$\mathrm{d}X_t = \mathrm{sgn}_0(X_t)\mathrm{d}W_t,$$

and X is a Brownian motion, moreover

$$\int_0^t \mathrm{sgn}_0(X_t)\mathrm{d}X_t = \int_0^t \mathrm{sgn}_0(X_t)^2\mathrm{d}W_t = W_t - \int_0^t \mathbb{1}_{X_s=0}\mathrm{d}W_s = W_t$$

since $[\int_0^\cdot \mathbb{1}_{X_s=0}\mathrm{d}W_s]_T = \int_0^T \mathbb{1}_{X_s=0}\mathrm{d}s = 0$. By Ito-Tanaka's formula

$$|X_t| = \int_0^t \mathrm{sgn}_{-1}(X_s)\mathrm{d}X_s + L_t^{X,0}$$

where $L_t^{X,0}$ is the local time of X in 0, and this shows that

$$W_t = \int_0^t \mathrm{sgn}_0(X_t)\mathrm{d}X_t = \int_0^t \mathrm{sgn}_{-1}(X_s)\mathrm{d}X_s = |X_t| - L_t^{X,0}$$

and recalling that we have (since X is a martingale)

$$L_t^{X,0} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|X_s|<\varepsilon}\mathrm{d}s,$$

which implies that W is measurable wrt. the filtration generated by $|X|$. If we had a strong solution then we would have that $\mathscr{F}_t^X \subseteq \mathscr{F}_t^W \subseteq \mathscr{F}_t^{|X|}$ which is not possible because you cannot recover the sign of a Brownian motion only knowing its absolute value.

So there are no strong solution and a consequence there is no pathwise uniqueness (by Yamada–Watanabe).

Exercise 1. Prove that if B is a Brownian motion, then we have the relation $L_t^{|B|,0} = 2L_t^{B,0}$.

We go back to the equation

$$R_t = |B_t| = \underbrace{\int_0^t \mathrm{sgn}_{-1}(B_s)\mathrm{d}B_s}_{\beta_t} + L_t$$

we want to show that in this equation both R, L are functions of the Brownian motion β_t which we think as given, according to the following definition

Definition 1. *(Reflected SDE) The family (X, ℓ, W) is a weak solution of the one dimensional reflected SDE*

$$\mathrm{d}X_t = \mathrm{d}W_t + \mathrm{d}\ell_t$$

if W is a Brownian motion, ℓ a continuous positive non-decreasing process and X a continuous positive process such that

$$\int_0^\infty \mathbb{1}_{X_s>0}\mathrm{d}\ell_s = 0.$$

The solution is strong if (X, ℓ) is adapted to the noise W.

Therefore (R, L, β) is a weak solution of this reflected SDE. We will need the followin analysis lemma (we use $\mathbb{R}_+ = \mathbb{R}_{\geqslant 0}$)

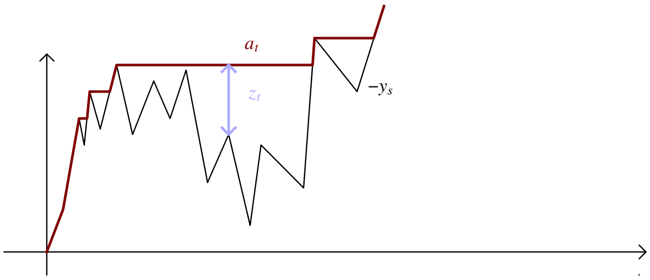
Lemma 2. *(Skorokhod lemma) Let $y \in C(\mathbb{R}_+; \mathbb{R})$ such that $y(0) \geqslant 0$. There exists a unique pair (z, a) with $z \in C(\mathbb{R}_+; \mathbb{R}_+)$ and $a \in C(\mathbb{R}_+; \mathbb{R}_+)$ with a non-decreasing, $a(0) = 0$, such that*

a) $z_t = y_t + a_t$

b) $\int_0^\infty \mathbb{1}_{z_s>0} \mathrm{d}a_s = 0$.

Moreover

$$a(t) = \sup_{s\in[0,t]} (y_s)_- = \sup_{s\in[0,t]} (-y_s \vee 0). \tag{2}$$



Proof. Exercise prove that if we let a as in eq. (2) then *a),b)* are satisfied, this settles the existence part. As for uniqueness we assume that both (z,a) and (z',a') are two solutions of this problem. Then $y_t = z_t - a_t = z_t' - a_t'$ so we have $z_t - z_t' = a_t - a_t'$ so $h_t = z_t - z_t'$ is of bounded variation (as a difference of two increasing functions) and we can write (by Ito formula)

$$\mathrm{d}(z_t - z_t')^2 = 2\int_0^t (z_s - z_s')\mathrm{d}(z_s - z_s') = 2\int_0^t (z_s - z_s')\mathrm{d}(a_s - a_s') = 2\int_0^t (z_s - z_s')\mathrm{d}a_s - 2\int_0^t (z_s - z_s')\mathrm{d}a_s'$$

$$= 2\int_0^t (-z_s')\mathrm{d}a_s - 2\int_0^t (z_s)\mathrm{d}a_s' \leqslant 0$$

where we used that $\int_0^t z_s \mathrm{d}a_s = \int_0^t z_s' \mathrm{d}a_s' = 0$ and that $z_s, z_s' \geqslant 0$. So $h_t^2 \geqslant 0$ is decreasing and since $h_0 = 0$ we have that $h_t = 0$ for any t. This establish uniqueness. □

As a consequence of this lemma we have that the reflected SDE has a unique solution in law (and pathwise) which is given therefore by

$$\ell_t = \sup_{s\in[0,t]} (-W_s)_+ = \sup_{s\in[0,t]} (-W_s) = S_t^{-W} \qquad X_t = W_t + \ell_t$$

where we note $S_t^W = \sup_{s\leqslant t} W_t$ and the solution is strong.

Definition 3. *We call the process X the reflected Brownian motion*

We deduce as a consequence that if we consider

$$R_t = |B_t| = \underbrace{\int_0^t \mathrm{sgn}_{-1}(B_s)\mathrm{d}B_s}_{\beta_t} + L_t$$

then we have

$$L_t = \sup_{s\in[0,t]} (-\beta_s)_+ = \sup_{s\in[0,t]} (-\beta_s) = S_t^{-\beta}.$$

From this we deduce

Theorem 4.

$$\mathrm{Law}(|B|, L) = \mathrm{Law}(\beta + L, L) = \mathrm{Law}(\beta + S^{-\beta}, S^{-\beta}) = \mathrm{Law}(S^W - W, S^W)$$

where W here is a generic Brownian motion. This formula allows to compute the joint law of the supremum S^W of a Brownian motion W together with the Brownian motion, in terms of the law of the reflected Brownian motion R.

Remark 5. Some of the utility of this relation come from the fact that it implies that

$$\mathrm{Law}(|B_t|, L_t) = \mathrm{Law}(S_t^W - W_t, S_t^W)$$

and that by the reflection principle one can compute explicitly the law $\mathrm{Law}(S_t^W - W_t, S_t^W)$, or moreover that

$$\mathrm{Law}(|B|) = \mathrm{Law}(S^W - W)$$

which given informations on the supremum S^W in terms of the modulus of another Brownian motion.

New chapter

1 Brownian martingale representation theorem

We concentrate now in the study of the probability space generated by a Brownian motion (maybe multidimensional, taking values in $\mathbb{R}^n$). We assume in this part that $(\Omega, \mathscr{F}, (\mathscr{F}_t)_t, \mathbb{P})$ is the canonical n-dimensional Wiener space, i.e. $\Omega = \mathscr{C}^n = C(\mathbb{R}_+, \mathbb{R}^n)$, $X_t(\omega) = \omega(t)$, $\mathbb{P}$ is the law of the Brownian motion and $(\mathscr{F}_t)_{t \geqslant 0}$ is the right continuous $\mathbb{P}$-completed filtration generated by the canonical process $(X_t)_{t \geqslant 0}$ in particular we have $\mathscr{F}_\infty = \mathscr{F} = \mathscr{B}(\Omega)$. This is called a Brownian probability space.

Theorem 6. *Let $\Phi \in L^2(\Omega, \mathscr{F}, \mathbb{P})$, then there exists a unique predictable process $F \in L^2_{\mathscr{P}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^n)$ such that*

$$\Phi(X) = \mathbb{E}[\Phi(X)] + \sum_{k=1}^{n} \int_0^\infty F_s^{(k)}(X) \mathrm{d}X_s^{(k)}.$$

This theorem says that any mean zero L^2 random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ can be written as a stochastic integral wrt. the Brownian motion. It will have as a consequence that any martingale on $(\Omega, \mathscr{F}, \mathbb{P})$ is a stochastic integral wrt. to (the given) Brownian motion and therefore it has a continuous modification. This rules out the possibility that martingales on a Brownian probability space has jumps, “informations comes in in a continuous way”.

Remark 7. This theorem is connected with something called “Malliavin calculus” in which the function F represents a kind of derivative of Φ wrt. X_s. And with the fact that iterated stochastic integrals are dense in $L^2(\Omega, \mathscr{F}, \mathbb{P})$. They play the role of orthogonal polynomials in the Hilbert space $L^2(\Omega, \mathscr{F}, \mathbb{P})$.

Deadline for the next sheet is next friday as writted on eCampus.