Lecture 18 – 2020.06.23 – 12:15 via Zoom

Brownian martingale representation theorem

We concentrate now in the study of the probability space generated by a Brownian motion (maybe multidimensional, taking values in $\mathbb{R}^d$). We assume in this part that $(\Omega, \mathscr{F}, (\mathscr{F}_t)_t, \mathbb{P})$ is the canonical d-dimensional Wiener space, i.e. $\Omega = \mathscr{C}^d = C(\mathbb{R}_+, \mathbb{R}^d)$, $X_t(\omega) = \omega(t)$, $\mathbb{P}$ is the law of the Brownian motion and $(\mathscr{F}_t)_{t \geqslant 0}$ is the right continuous $\mathbb{P}$-completed filtration generated by the canonical process $(X_t)_{t \geqslant 0}$ in particular we have $\mathscr{F}_\infty = \mathscr{F} = \overline{\mathscr{B}(\Omega)}^{\mathbb{P}}$. This is called a Brownian probability space.

Theorem 1. *Let $\Phi \in L^2(\Omega, \mathscr{F}, \mathbb{P})$, then there exists a unique predictable process $F \in L^2_{\mathscr{P}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^n)$ such that*

$$\Phi = \mathbb{E}[\Phi] + \sum_{k=1}^d \int_0^\infty F_s^{(k)} \mathrm{d} X_s^{(k)}.$$

This theorem says that any mean zero L^2 random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ can be written as a stochastic integral wrt. the Brownian motion. It will have as a consequence that any martingale on $(\Omega, \mathscr{F}, \mathbb{P})$ is a stochastic integral wrt. to (the given) Brownian motion and therefore it has a continuous modification. This rules out the possibility that martingales on a Brownian probability space has jumps, "informations comes in in a continuous way".

Remark 2. This theorem is connected with something called "Malliavin calculus" in which the function F represents a kind of derivative of Φ wrt. X_s. And with the fact that iterated stochastic integrals are dense in $L^2(\Omega, \mathscr{F}, \mathbb{P})$. They play the role of orthogonal polynomials in the Hilbert space $L^2(\Omega, \mathscr{F}, \mathbb{P})$.

We will give a "Markovian" proof. In the next exercise sheet you will be asked to give a "Gaussian" proof.

We need this technical lemma.

Lemma 3. *Let $p \geqslant 1$ and $\mathscr{C} \subseteq L^\infty(\Omega, \mathscr{F}, \mathbb{P})$ be the algebra generated by the random variables*

$$\Phi^\alpha(f) := \int_0^\infty e^{-\alpha t} f(X_t) \mathrm{d} t$$

where $\alpha > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$ (smooth and compact support). Then $\mathscr{C}$ is dense in $L^p(\Omega, \mathscr{F}, \mathbb{P})$.

The interest of this algebra of functions is that it behaves *nicely* wrt. Markov processes. (The proof really uses only the continuity of the trajectories of X and the fact that $\mathscr{F}$ is the filtration generated by X.

Proof. (of Theorem 1) If $F \in \mathscr{C}$ we can give an explicit martingale representation because conditional expectations of elements in $\mathscr{C}$ can be computed explicitly. Take for example $\Phi^\alpha(f)$, then we have by the Markov property

$$\mathbb{E}[\Phi^\alpha(f) | \mathscr{F}_t] = \mathbb{E}\left[\int_0^\infty e^{-\alpha s} f(X_s) \mathrm{d} s \middle| \mathscr{F}_t\right] = \int_0^t e^{-\alpha s} f(X_s) \mathrm{d} s + \int_t^\infty e^{-\alpha s} \underbrace{\mathbb{E}[f(X_s) | \mathscr{F}_t]}_{(P_{s-t} f)(X_t)} \mathrm{d} s$$

$$= \int_0^t e^{-\alpha s} f(X_s) \mathrm{d} s + \int_t^\infty e^{-\alpha s} (P_{s-t} f)(X_t) \mathrm{d} s$$

$$= \int_0^t e^{-\alpha s} f(X_s) \mathrm{d} s + e^{-\alpha t} \underbrace{\int_0^\infty e^{-\alpha s} P_s f(X_t) \mathrm{d} s}_{=: U^\alpha(f)(X_t)}$$

where we let $U^\alpha f(x) \coloneqq \int_0^\infty e^{-\alpha t} P_t f(x) \mathrm{d}x$ for any $\alpha > 0$ (the resolvent operator) and $f \in C(\mathbb{R}^d)$ and with P_t the transition operator for the Brownian motion:

$$P_t f(x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{-|x-y|^2/t^2} \mathrm{d}y.$$

Recall that a generic element of $\mathscr{C}$ is a finite linear combination of monomials of the form

$$\prod_{i=1}^n \Phi^{\alpha_i}(f_i)$$

for some $\alpha_1, \ldots, \alpha_n > 0$ and $f_1, \ldots, f_n \in C_0^\infty(\mathbb{R}^d)$. This can be written as (where S_n is the set of permutations of n elements, and $t \geqslant 0$ is arbitrary)

$$\begin{aligned}
\prod_{i=1}^n \Phi^{\alpha_i}(f_i) &= \sum_{\sigma \in S_n} \int_{0 < s_1 < \cdots < s_n} \left[\prod_{i=1}^n e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_1 \cdots \mathrm{d}s_n \\
&= \sum_{\sigma \in S_n} \int_{0 < s_1 < \cdots < s_n} \left[\prod_{i=1}^n (\mathbb{1}_{s_i \leqslant t} + \mathbb{1}_{s_i > t}) e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_1 \cdots \mathrm{d}s_n \\
&= \sum_{k=0}^n \sum_{\sigma \in S_n} \int_{0 < s_1 < \cdots < s_n} \mathbb{1}_{s_k \leqslant t} \mathbb{1}_{s_{k+1} > t} \left[\prod_{i=1}^n e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_1 \cdots \mathrm{d}s_n \\
&= \sum_{k=0}^n \sum_{\sigma \in S_n} \int_{0 < s_1 < \cdots < s_k < t} \left[\prod_{i=1}^k e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_1 \cdots \mathrm{d}s_k \int_{t \leqslant s_{k+1} < s_n} \left[\prod_{i=k+1}^n e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n \\
&= \sum_{k=0}^n \sum_{\sigma \in S_n} V_t^{\sigma,k}(X) \int_{t \leqslant s_{k+1} < s_n} \left[\prod_{i=k+1}^n e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n
\end{aligned}$$

where we use the convention that $s_0 = 0$ and $s_{n+1} = +\infty$ and where we let

$$V_t^{\sigma,k}(X) = \int_{0 < s_1 < \cdots < s_k < t} \left[\prod_{i=1}^k e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_1 \cdots \mathrm{d}s_k.$$

A computation using the Markov property inductively gives

$$\begin{aligned}
&\mathbb{E}\left[\int_{t \leqslant s_{k+1} < s_n} \left[\prod_{i=k+1}^n e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n \middle| \mathscr{F}_t \right] \\
&= \mathbb{E}\left[\int_{t \leqslant s_{k+1} < s_{n-1}} \left[\prod_{i=k+1}^{n-1} e^{-\alpha_{\sigma(i)} s_i} f_{\sigma(i)}(X_{s_i}) \right] e^{-\alpha_{\sigma(n)} s_{n-1}} U^{\alpha_{\sigma(n)}} f_{\sigma(n)}(X_{s_{n-1}}) \mathrm{d}s_{k+1} \cdots \mathrm{d}s_{n-1} \middle| \mathscr{F}_t \right] \\
&= e^{-\alpha(\sigma,k)t} U^{\alpha(\sigma,k)} (f_{\sigma(k+1)} U^{\alpha(\sigma,k+1)} (f_{\sigma(k+2)} \cdots (f_{\sigma(n-1)} U^{\alpha(\sigma,n-1)} (f_{\sigma(n)})))) (X_t) \\
&= e^{-\alpha(\sigma,k)t} U^{\alpha(\sigma,k)} (H^{\sigma,k})(X_t)
\end{aligned}$$

where $\alpha(\sigma, k) = \alpha_{\sigma(k+1)} + \alpha_{\sigma(k+2)} + \cdots + \alpha_{\sigma(n)}$ and

$$H^{\sigma,k}(x) \coloneqq f_{\sigma(k+1)}(x) U^{\alpha(\sigma,k+1)} (f_{\sigma(k+2)} \cdots (f_{\sigma(n-1)} U^{\alpha(\sigma,n)} (f_{\sigma(n)}))) (x) = f_{\sigma(k+1)}(x) U^{\alpha(\sigma,k+1)} (H^{\sigma,k+1})(x)$$

$$H^{\sigma,n}(x) \coloneqq f_{\sigma(n)}(x).$$

We conclude that

$$M_t = \mathbb{E}\left[\prod_{i=1}^n \Phi^{\alpha_i}(f_i) \middle| \mathscr{F}_t \right] = \sum_{k=0}^n \sum_{\sigma \in S_n} V_t^{\sigma,k}(X) e^{-\alpha(\sigma,k)t} U^{\alpha(\sigma,k)} (H^{\sigma,k})(X_t). \tag{1}$$

This formula shows that the martingale $(M_t)_{t\geqslant 0}$ is continuous in $t\in\mathbb{R}$ since this is so for the the r.h.s. since $V_t^{\sigma,k}(X)$ is an integral and therefore continuous in t and $U^{\alpha(\sigma,k)}(H^{\sigma,k})(x)$ a smooth function of x. Note that $t\mapsto V_t^{\sigma,k}(X)e^{-\alpha(\sigma,k)t}$ is a bounded variation process. So the only contributions to the martingale M_t must come from the processes $t\mapsto U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t)$. By Ito formula we have

$$\mathrm{d}U^{\alpha(\sigma,k)}(H^{\sigma,k})(X_t)=\nabla(U^{\alpha(\sigma,k)}(H^{\sigma,k}))(X_t)\mathrm{d}X_t+\text{bounded variation part}$$

we do not care about the bounded variation part since it has to cancel with the bounded variation part coming from $t\mapsto V_t^{\sigma,k}(X)e^{-\alpha(\sigma,k)t}$ (maybe, as an exercise, you can check it). By equating the two continuous local martingales on the l.h.s. and r.h.s. of eq. (1) we deduce that

$$M_t-M_0=\int_0^t F_s\cdot\mathrm{d}X_s$$

where

$$F_s:=\sum_{k=0}^n\sum_{\sigma\in S_n}V_s^{\sigma,k}(X)e^{-\alpha(\sigma,k)s}\nabla(U^{\alpha(\sigma,k)}(H^{\sigma,k}))(X_s).$$

By taking $t\to\infty$ this shows that (by martingale convegence theorem in L^2)

$$\prod_{i=1}^n\Phi^{\alpha_i}(f_i)=\mathbb{E}\left[\prod_{i=1}^n\Phi^{\alpha_i}(f_i)\right]+\int_0^\infty F_s\cdot\mathrm{d}X_s$$

indeed note that by Ito isometry

$$\left(\mathbb{E}\left[\prod_{i=1}^n\Phi^{\alpha_i}(f_i)\right]\right)^2+\mathbb{E}\left[\left(\int_0^\infty F_s\cdot\mathrm{d}X_s\right)^2\right]=\mathbb{E}\left[\left(\prod_{i=1}^n\Phi^{\alpha_i}(f_i)\right)^2\right]<\infty.$$

Any $\Phi\in\mathscr{C}$ can be written as a stochastic integral wrt. Brownian motion plus a constant.

For general $\Phi\in L^2(\Omega,\mathscr{F},\mathbb{P})$ we can choose a sequence $(\Phi_n)_{n\geqslant 1}\subset\mathscr{C}$ such that $\Phi_n\to\Phi$ in L^2. Now let $M_t^n:=\mathbb{E}[\Phi^n|\mathscr{F}_t]$ and $M_t=\mathbb{E}[\Phi|\mathscr{F}_t]$.

By the previous step we know there exists adapted functions $F^n\in L^2_{\mathscr{P}}(\mathbb{R}_+\times\Omega)$ such that

$$M_t^n=\mathbb{E}[\Phi^n]+\int_0^t F_s^n\mathrm{d}X_s,$$

therefore by Ito isometry and $n,m\geqslant 1$

$$\mathbb{E}[(M_t^n-M_t^m)^2]=\mathbb{E}[[M^n-M^m]_t]=\mathbb{E}\int_0^t|F_s^n-F_s^m|^2_{\mathbb{R}^d}\mathrm{d}s,\qquad t\geqslant 0.$$

therefore

$$\mathbb{E}\int_0^\infty|F_s^n-F_s^m|^2_{\mathbb{R}^d}\mathrm{d}s\leqslant\sup_t\mathbb{E}[(M_t^n-M_t^m)^2]\leqslant\mathbb{E}\left[\sup_{t\geqslant 0}(M_t^n-M_t^m)^2\right]=o_{n,m}(1)$$

By martingale convergence theorem we have that $M_t^n\to M_t$ a.s. and in L^2 and by Doob's maximal inequality this convegence is uniform in t (here we need that the filtration is right-continuous). This implies also that $(F^n)_{n\geqslant 1}$ is a Cauchy sequence in $L^2_{\mathscr{P}}(\mathbb{R}_+\times\Omega)$ which is complete therefore there exists a unique limit $F=\lim_n F^n\in L^2_{\mathscr{P}}(\mathbb{R}_+\times\Omega)$ and from this we get that

$$M_t=\mathbb{E}[\Phi]+\int_0^t F_s\mathrm{d}X_s.$$

By taking $t\to\infty$ and using L^2 convergence and $M_t\to\mathbb{E}[\Phi|\mathscr{F}_\infty]=\Phi$ in L^2 (because $\mathscr{F}_\infty=\mathscr{F}$) we obtain that there exists $F\in L^2_{\mathscr{P}}(\mathbb{R}_+\times\Omega)$ such that

$$\Phi=\mathbb{E}[\Phi]+\int_0^\infty F_s\mathrm{d}X_s.$$

In general there is no easy formula for F. □

Corollary 4. *All local martingales in a Brownian probability space are continuous.*

Proof. Exercise. □

Applications of the martingale representation theorem

a) Mathematical finance: if you model the evolution of stock prices with the probability space generated by a multidimensional Brownian motion X then any "contract" Φ can be expressed as

$$\Phi = \mathbb{E}[\Phi] + \int_0^\infty F_s \mathrm{d}X_s$$

which means that we can replicate the contract by trading the underlying assets X using the strategy given by F (if we are able to compute or approximate F). The strategy F (which is a vector $(F^1, \ldots, F^d)$) has to be interpreted as follows: F^k is the number of stocks of the asset k which one has to acquire at the beginning of every "infinitesimal" trading round.

b) Study of the entropy $H(\mathbb{Q}|\mathbb{P})$ of two measures $\mathbb{P}, \mathbb{Q}$ on the Brownian probability space with application to the estimation of averages of functionals and to small noise large deviations of diffusion, i.e. investigate the behaviour of the law μ^ε of the solution of the SDE

$$\mathrm{d}X_t^\varepsilon = b(X_t^\varepsilon)\mathrm{d}t + \varepsilon\sigma(X_t^\varepsilon)\mathrm{d}W_t$$

as $\varepsilon \to 0$.

c) Backward SDEs (BSDE): this is a class of stochastic differential equations with final condition (instead of initial condition). Let Φ be a given random variable which is $\mathscr{F}_T$ measurable for given $T > 0$ (deterministic) the solution to a BSDE with *driver* $f(t, y, z)$ is a pair (Y, Z) of adapted processes such that

$$-\mathrm{d}Y_t = f(t, Y_t, Z_t)\mathrm{d}t + Z_t \mathrm{d}W_t, \qquad t \in [0, T]$$

and $Y_T = \Phi$, where $(W_t)_{t \geqslant 0}$ it is an adapted Brownian motion and $t \mapsto f(t, y, z)$ an adapted process. This kind of equations has application in finance but also applications in the representations of solutions to non-linear PDEs (very much like SDE can represent solutions to certain classes of linear PDEs, e.g. via Feynman-Kac formula).