Lecture 2 – 23.04.2020 – 12:15 via Zoom

Stochastic differential equations

Existence, uniqueness, various notions thereof, relations between such notions.

Setting. Probability space $(\Omega, \mathscr{F}, \mathbb{P})$, filtration $(\mathscr{F}_t)_{t \geqslant 0}$ right-continuous, completed.

Definition 1. *A **(weak) solution** of the SDE in* $\mathbb{R}^n$

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t \in [0, T]$$

$$X_0 = x \in \mathbb{R}^n$$

is a pair of adapted processes (X, B) *where* $(B_t)_{t \geqslant 0}$ *is a m-dimensional Brownian motion and* b, σ *are coefficients* $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m; \mathbb{R}^n)$ *such that almost surely*

$$\int_0^t |b(X_s)|\mathrm{d}s < \infty, \quad \int_0^t \mathrm{Tr}(\sigma(X_s)\sigma(X_s)^T)\mathrm{d}s < \infty, \qquad t \in [0, T]$$

and that

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s, \qquad t \in [0, T].$$

Note: a weak solution is really the data $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t \geqslant 0}, X, B)$.

$\sigma = (\sigma_\alpha)_{\alpha = 1, \ldots, m}$ family of vector-fields $\sigma_\alpha: \mathbb{R}^n \to \mathbb{R}^n$ (this is the right point of view on manifolds)
Control theory point of view:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sum_{\alpha = 1}^m \sigma_\alpha(X_t)\mathrm{d}B_t^\alpha.$$

$$\sum_{\alpha = 1}^m \int_0^t |\sigma_\alpha(X_t)|^2 \mathrm{d}s < \infty.$$

Definition 2. *A **strong solution** to the SDE above is a weak solution such that X is adapted to the* $\mathbb{P}$*-completed filtration* $(\mathscr{F}_t^B)_{t \geqslant 0}$ *generated by B,* $\mathscr{F}_t^B := \overline{\sigma(B_s: s \in [0, t])}^{\mathbb{P}}$.

As a consequence

$$X_t \hat{\in} \mathscr{F}_t \Rightarrow X_t(\omega) = \Phi_t((B_s(\omega))_{s \in \in [0, t]})$$

$\Phi_t: C([0, t]; \mathbb{R}^m) \to \mathbb{R}^n$. While in general we could have

$$X_t(\omega) = \Phi_t((B_s(\omega))_{s \in \in [0, t]}, N(\omega)).$$

$$X = \Psi(B), \qquad \Psi: C([0, T]; \mathbb{R}^m) \to C([0, T]; \mathbb{R}^n), \quad \text{measurable}$$

Definition 3. *An SDE has* ***uniqueness in law*** *iff two solutions* $(\Omega,\mathscr{F},\mathbb{P},(\mathscr{F}_t)_{t\geqslant 0},X,B)$, $(\Omega',\mathscr{F}',\mathbb{P}',(\mathscr{F}_t')_{t\geqslant 0},X',B')$ *are such that*

$$\mathrm{Law}_{\mathbb{P}}(X)=\mathrm{Law}_{\mathbb{P}'}(X')\in\Pi(C([0,T];\mathbb{R}^n),\mathscr{B}(C([0,T];\mathbb{R}^n)))$$

Definition 4. *An SDE has* ***pathwise uniqueness*** *if for any two weak solutions X,X' defined on the same filt. prob. space and with the* ***same*** *BM B we have that they are indistinguishable, i.e.*

$$\mathbb{P}(\exists t\in[0,T]: X_t\neq X_t')=0.$$

Some examples of all the possible situations

Example 5. [No existence] The following SDE on $\mathbb{R}$ has no weak solution

$$\mathrm{d}X_t=-\frac{1}{2X_t}\mathbb{1}_{X_t\neq 0}\mathrm{d}t+\mathrm{d}B_t,\quad X_0=0. \tag{1}$$

Ito formula

$$X_t^2=2\int_0^t X_s\mathrm{d}X_s+\int_0^t\mathrm{d}s=-\int_0^t\mathbb{1}_{X_s\neq 0}\mathrm{d}s+2\int_0^t X_s\mathrm{d}B_s+\int_0^t\mathrm{d}s=\int_0^t\mathbb{1}_{X_s=0}\mathrm{d}s+2\int_0^t X_s\mathrm{d}B_s.$$

Since $[X]_t=t$ then the occupation time formula (which we assume for now, we will go back to this when discussing Tanaka's formula) we have

$$\int_0^t\mathbb{1}_{X_s=0}\mathrm{d}s=0.$$

Therefore $(X_t^2)_t$ is a local martingale, which is positive and such that $X_0^2=0\Rightarrow X_t=0$ for all $t\in[0,T]$. But $X_t=0$ is not a solution to the SDE (1).

Example 6. [No strong sol, nor pathwise uniqueness, ∃weak solutions, uniqueness in law] Tanaka's SDE:

$$\mathrm{d}X_t=\mathrm{sgn}(X_t)\mathrm{d}B_t,\quad X_0=0. \tag{2}$$

Example 7. [No uniqueness, ∃strong]

$$\mathrm{d}X_t=\mathbb{1}_{X_t\neq 0}\mathrm{d}B_t,\quad X_0=0.$$

The process $X_t=0$ is a solution but also the process $X_t=B_t$ is a solution, indeed in this second case we have $X_t-B_t=-\int_0^t\mathbb{1}_{X_s=0}\mathrm{d}B_s$ and this process has zero quadratic variation almost surely:

$$[X-B]_t=\int_0^t\mathbb{1}_{X_s=0}\mathrm{d}s=0$$

by the *occupation time formula* since $\mathrm{d}[X]_t\ll\mathrm{d}t$. No pathwise-!, law-!. Assume on the probability space there is also a Bernoulli variable ξ (e.g. independent of B) assume $\mathscr{F}_0\supseteq\sigma(\xi)$ and let

$$X_t(\omega)=\begin{cases}0 & \text{if } \xi(\omega)=+1\\ B_t(\omega) & \text{if } \xi(\omega)=0\end{cases}$$

This solution is not strong.

Example 8. [No strong sol. and no uniq.]

$$\mathrm{d}X_t = \mathbb{1}_{X_t \neq 1} \operatorname{sgn}(X_t) \mathrm{d}B_t, \quad X_0 = 0.$$

Here there exists weak solutions, no pathwise uniq., no strong solutions, no uniqueness in law. Indeed the Tanaka example Y is a solution but also $Z_t = Y_{t \wedge \tau}$ where $\tau = \inf\{t \geqslant 0 : Y_t = 1\}$.

Theorem 9. (Yamada–Watanabe) *Weak existence+pathwise uniqueness* $\Rightarrow$ *strong existence*

Theorem 10. *pathwise uniqueness* $\Rightarrow$ *uniqueness in law*

Theorem 11. *(Cerny) Strong existence+uniquess in law* $\Rightarrow$ *pathwise uniqueness*

Theorem 12. *(Cerny) Uniqueness in law implies uniqueness of the law of the weak solution* (X, B)

We are going to sketch the proofs of these facts.

Weak existence is usually obtained via approximations, apriori estimates and compactness arguments. Pathwise uniqueness is done by direct comparison of two solutions.

Proof. Of Theorem 10. Take two solutions $(\Omega, \mathbb{P}, \mathscr{F}, X, B)$, $(\Omega', \mathbb{P}', \mathscr{F}', X', B')$ we know pathwise uniqueness and we want to deduce $\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{P}'}(X')$. It would be easy if $(\Omega, \mathscr{F}) = (\Omega', \mathscr{F}')$ and $B = B'$ since then pathwise uniqueness applies and $X = X'$ from which follows that their laws are the same. Let almost surely

$$\rho_{B(\omega)}(A) = \mathbb{P}(X \in A | B)(\omega), \qquad \rho'_{B'(\omega')}(A) = \mathbb{P}'(X' \in A | B')(\omega'), \qquad A \in \mathscr{B}(\mathscr{C}^n)$$

with $\mathscr{C}^n = C(\mathbb{R}_+; \mathbb{R}^n)$. Both ρ, ρ' are regular conditional probabilities, i.e. probability kernels

$$\mathscr{C}^m \to \Pi(\mathscr{C}^n, \mathscr{B}(\mathscr{C}^n)).$$

This is possible since $\mathscr{C}^m$ is Polish. We can define a probabily measure $\mathbb{Q}$ on the filtered measure space $\tilde{\Omega} = \mathscr{C}^n \times \mathscr{C}^n \times \mathscr{C}^m$ with canonical process $(X_t, Y_t, B_t) : \tilde{\Omega} \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ given by

$$\mathbb{Q}(\mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3) = \rho_{\omega_3}(\mathrm{d}\omega_1) \rho'_{\omega_3}(\mathrm{d}\omega_2) \mu(\mathrm{d}\omega_3)$$

where $\mu \in \Pi(\mathscr{C}^m)$ is the law of the Brownian motion in $\mathbb{R}^m$. Then easy to check that

$$\mathrm{Law}_{\mathbb{Q}}(X, B) = \mathrm{Law}_{\mathbb{P}}(X, B), \qquad \mathrm{Law}_{\mathbb{Q}}(Y, B) = \mathrm{Law}_{\mathbb{P}'}(X', B').$$

Technical point (that we will not prove here): the process $(\tilde{\Omega}, \mathbb{Q}, X, B)$ is a weak solution and $(\tilde{\Omega}, \mathbb{Q}, Y, B)$ is also a weak solution. Assuming this, by pathwise uniqueness we have that $X = Y$ almost surely which implies that

$$\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{Q}}(X) = \mathrm{Law}_{\mathbb{Q}}(Y) = \mathrm{Law}_{\mathbb{P}'}(X')$$

that is uniqueness in law. □

Proof. Of Theorem 9 (Yamada–Watanabe). We want to prove that there exists $\Phi: \mathscr{C}^m \to \mathscr{C}^n$ such that letting $Z = \Phi(B)$ we have that (Z, B) is a solution to the SDE. In this case we should have that its law is given by

$$\mathbb{P}((Z,B) \in (\mathrm{d}\omega_1 \times \mathrm{d}\omega_2)) = \delta_{\Phi(\omega_2)}(\mathrm{d}\omega_1)\,\mu(\mathrm{d}\omega_2), \qquad \omega_1 \in \mathscr{C}^n, \omega_2 \in \mathscr{C}^m$$

But the the previous argument give us that any two weak solutions have the same joint distributions, that is $\mathrm{Law}_{\mathbb{P}}(X, B) = \mathrm{Law}_{\mathbb{P}'}(X', B')$. In this case we would have also

$$\delta_{\Phi(\omega_2)} = \rho_{\omega_2} = \rho'_{\omega_2}, \qquad \omega_2 \in \mathscr{C}^m.$$

But to prove this, namely that ρ and ρ' are δ measures we observe that pathwise uniquenes above give us

$$1 = \mathbb{Q}(X = Y) = \int_{\tilde{\Omega}} \mathbb{1}_{\omega_1 = \omega_2} \rho_{\omega_3}(\mathrm{d}\omega_1)\,\rho_{\omega_3}(\mathrm{d}\omega_2)\,\mu(\mathrm{d}\omega_3),$$

by Fubini this implies

$$\int_{\mathscr{C}^n} \mathbb{1}_{\omega_1 = \omega_2} \rho_{\omega_3}(\mathrm{d}\omega_1) = 1, \qquad \text{for } \rho_{\omega_3}(\mathrm{d}\omega_2)\,\mu(\mathrm{d}\omega_3)\text{-a.e. } (\omega_2, \omega_3).$$

Therefore $\rho_{\omega_3}(\mathrm{d}\omega_1) = \delta_{\omega_2}(\mathrm{d}\omega_1)$ for almost every (ω_2, ω_3). From this is easy to deduce (exercise) that there exists Φ such that

$$\rho_{\omega_3}(\mathrm{d}\omega_1) = \delta_{\Phi(\omega_3)}(\mathrm{d}\omega_1).$$

Now one has to prove a technical lemma which says that the map Φ can be chosen to be adapted. □
