Lecture 20 – 2020.06.30 – 12:15 via Zoom

Exam: first oral exam from 27/7-1/8. second oral exam mid september 14/9-25/9.

Boué–Dupuis formula (continued)

We assume that $(\Omega, \mathscr{F}, (\mathscr{F}_t)_t, \mathbb{P})$ is the canonical d-dimensional Wiener space, i.e. $\Omega = \mathscr{C}^d = C(\mathbb{R}_+, \mathbb{R}^d)$, $X_t(\omega) = \omega(t)$, $\mathbb{P}$ is the law of the Brownian motion and $(\mathscr{F}_t)_{t \geqslant 0}$ is the filtration generated by the canonical process $(X_t)_{t \geqslant 0}$ in particular we have $\mathscr{F}_\infty = \mathscr{F} = \mathscr{B}(\Omega)$. We will also use the notation μ for the Wiener measure $\mathbb{P}$.

Recall this lemma proven in the last lecture

Lemma 1. *Let ν be a probability measure which is absolutely continuous wrt. μ with density Z such that $Z \in \mathscr{C}$ (defined last week) and $Z \geqslant \delta$ for some $\delta > 0$. Let us call $\mathscr{S}_\mu \subseteq \Pi(\Omega)$ the set of all such measures. Then under $\nu \in \mathscr{S}_\mu$ the canonical process X is a strong solution of the SDE*

$$\mathrm{d}X_t = u_t(X)\mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0$$

where W is a ν-Brownian motion and u a drift such that

$$\|u_t(x) - u_t(y)\| \leqslant L \|x - y\|_{C([0,t];\mathbb{R}^d)} \qquad x, y \in \Omega \tag{1}$$

for some finite constant L. Moreover

$$H(\nu|\mu) = \frac{1}{2} \mathbb{E}_\nu \|u(X)\|_{\mathbb{H}}^2.$$

Recall that $\mathbb{H} = L^2(\mathbb{R}_+; \mathbb{R}^d)$.

We go on now to reconsider a last lemma before the actual proof.

Recall that

$$\log \mu[e^f] = \sup_\nu [\nu(f) - H(\nu|\mu)]$$

Lemma 2. *Let $f : \Omega \to \mathbb{R}$ which is measurable and bounded from below. Assume $\mu(e^f) < \infty$. For every $\varepsilon > 0$ there exists $\nu \in \mathscr{S}_\mu$ such that*

$$\log \mu[e^f] \leqslant \nu(f) - H(\nu|\mu) + \varepsilon.$$

If $\mu(e^f) = +\infty$ then there exist a sequence $(\nu_n) \subseteq \mathscr{S}_\mu$ such that

$$+\infty = \log \mu[e^f] = \sup_n (\nu_n(f) - H(\nu_n|\mu)).$$

Proof. We start by assuming that $\log \mu[e^f] < \infty$. By monotone convergence it is enough to consider only bounded functions f and moreover such that $\mu[e^f] = 1$. Indeed if f is bounded below I can introduce $f_n = (f \wedge n)$ which is now a bounded function for any n and if we prove the claim for bounded functions then we have that for any n and $\varepsilon > 0$ we have

$$\log \mu[e^{f_n}] \leqslant \nu_n(f_n) - H(\nu_n|\mu) + \varepsilon / 2$$

for some ν_n. But then we observe that $f_n \leqslant f$ so

$$\log \mu[e^{f_n}] \leqslant \nu_n(f) - H(\nu_n|\mu) + \varepsilon/2.$$

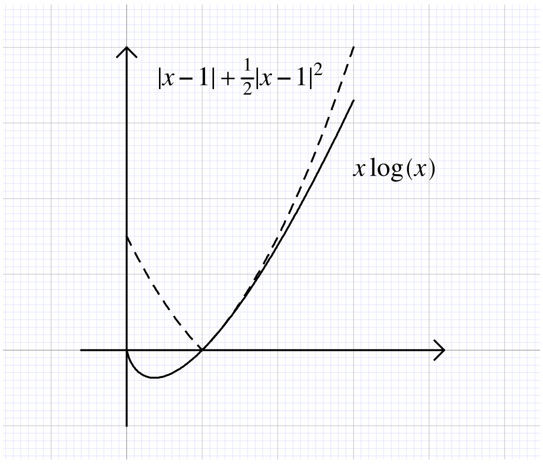
Moreover by monotone convergence we have $\log \mu[e^{f_n}] \to \log \mu[e^f]$. Then there exist n finite such that $\log \mu[e^f] \leqslant \log \mu[e^{f_n}] + \varepsilon/2$ and in this case we are done since

$$\log \mu[e^f] \leqslant \log \mu[e^{f_n}] + \varepsilon/2 \leqslant \nu_n(f) - H(\nu_n|\mu) + \varepsilon.$$

Note also that

$$\log \mu[e^{f-c}] - \nu(f-c) = \log \mu[e^f] - \nu(f)$$

so this shows that we can take c such that $\log \mu[e^{f-c}] = 0$, namely we can assume that f is such that $\mu[e^f] = 1$. Let $F = e^f$ and let ν be a probability measures on Ω. Note that



$$x\log(x) \leqslant |x-1| + \frac{1}{2}|x-1|^2, \qquad x \geqslant 0,$$

and using this we get

$$\mathrm{H}(\nu|\mu) - \nu(f) = \int_\Omega \left(\log\left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\omega)\right] - f(\omega)\right)\nu(\mathrm{d}\omega)$$

$$= \int_\Omega \left(\log\left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\omega)\right] - \log F(\omega)\right)\nu(\mathrm{d}\omega) = \int_\Omega \left(\log\left[\frac{1}{F(\omega)}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\omega)\right]\right)\nu(\mathrm{d}\omega)$$

$$= \int_\Omega \left(\log\left[\frac{1}{F(\omega)}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\omega)\right]\right)\left(\frac{1}{F(\omega)}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\omega)\right)F(\omega)\mu(\mathrm{d}\omega)$$

$$= \int_\Omega \left(\log\left[\frac{G(\omega)}{F(\omega)}\right]\right)\left(\frac{G(\omega)}{F(\omega)}\right)F(\omega)\mu(\mathrm{d}\omega)$$

where $G = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in \mathscr{C}$ since $\nu \in \mathscr{S}_\mu$. Using the inequality above we get

$$\mathrm{H}(\nu|\mu) - \nu(f) \leqslant \int_\Omega \left[\left|\frac{G}{F} - 1\right| + \frac{1}{2}\left|\frac{G}{F} - 1\right|^2\right] F(\omega)\mu(\mathrm{d}\omega) \leqslant \|F-G\|_{L^1(\mu)} + C_f\|F-G\|^2_{L^2(\mu)}$$

where the constant C_f depends only on the lower bound on f. Moreover $\|F-G\|_{L^1(\mu)} \leqslant \|F-G\|_{L^2(\mu)}$. This proves that $\mathrm{H}(\nu|\mu) - \nu(f)$ can be made as small as we want since $\mathscr{C}$ is dense in $L^2(\mu)$ and we can always find $G \in \mathscr{C}$ such that $G \geqslant \delta$ and $\|e^f - G\|_{L^2(\mu)} \leqslant \varepsilon$.

If $\log \mu[e^f] = +\infty$ the above argument allows to conclude the existence of the claimed sequence by using f_n as lower bound of f. □

Now we are going to complete the proof of

Theorem 3. *(Boué–Dupuis formula) For any function $f:\Omega\to\mathbb{R}$ measurable and bounded from below. We have*

$$\log \mathbb{E}_\mu[e^f] = \sup_{u\in\mathbb{H}} \mathbb{E}_\mu\left[f(X+I(u(X))) - \frac{1}{2}\|u(X)\|_{\mathbb{H}}^2\right]$$

where the supremum on the r.h.s. is taken wrt. all the predictable functions $u:\mathbb{R}_+\times\Omega\to\mathbb{R}$ such that

$$\|u\|_{\mathbb{H}}^2 = \int_0^\infty |u_s|^2 \mathrm{d}s < \infty, \qquad \mu - a.s. \tag{2}$$

and we write $u(\omega)=u(X(\omega))$ to stress the measurability wrt. the filtratrion $\mathscr{F}$ generated by X and where

$$I(u)(t) = \int_0^t u_s(X)\mathrm{d}s, \qquad t\geqslant 0.$$

We call a function u as above, a drift (wrt. μ).

Proof. We are going to prove that we have $\leqslant$ with an arbitrarily small loss ε and then that we have also the reverse inequality. Recall that we proved that if u is a drift and ν is the law of $X+I(u)$ then we have

$$H(\nu|\mu) \leqslant \frac{1}{2}\mathbb{E}_\mu[\|u(X)\|_{\mathbb{H}}^2]$$

then using this measure ν in the variational characterisation of $\log \mathbb{E}_\mu[e^f]$ we have

$$\log \mathbb{E}_\mu[e^f] = \sup_\rho (\rho(f) - H(\rho|\mu)) \geqslant \nu(f) - H(\nu|\mu)$$

$$\geqslant \nu(f) - \frac{1}{2}\mathbb{E}_\mu[\|u(X)\|_{\mathbb{H}}^2] = \mathbb{E}_\mu\left(f(X+I(u(X))) - \frac{1}{2}\|u\|_{\mathbb{H}}^2\right)$$

so we have one of the bounds because we can now optimize over all drifts u. In order to prove the reverse inequality we use the Lemma 2. Assume that $\log \mathbb{E}_\mu[e^f] < \infty$. For any $\varepsilon>0$ there exists $\nu\in\mathscr{S}_\mu$ satisfying

$$\log \mathbb{E}_\mu[e^f] - \varepsilon \leqslant \nu(f) - H(\nu|\mu)$$

Now recall by Lemma 1 under ν the canonical process satisfies the SDE $\mathrm{d}X = z(X)\mathrm{d}t + \mathrm{d}W$ for a "nice" drift z (which is Lipshitz) and a process W which is a Brownian motion under ν. This SDE has a unique strong solution, so we can write $X=\Phi(W)$ with some adapted functional Φ. Therefore we concolude that

$$X = W + I(z(X)) = W + I(u(W))$$

where we let $u(x) = z(\Phi(x))$ for all $x\in\Omega$. With this new expression we have that

$$\nu(f) = \mathbb{E}_\nu(f(X)) = \mathbb{E}_\nu(f(W+I(z(X)))) = \mathbb{E}_\nu(f(W+I(u(W)))) = \mathbb{E}_\mu(f(X+I(u(X))))$$

since $\mathrm{Law}_\nu(W) = \mathrm{Law}_\mu(X)$. Moreover we have also (for similar reasons)

$$H(\nu|\mu) = \frac{1}{2}\mathbb{E}_\nu\|z(X)\|_{\mathbb{H}}^2 = \frac{1}{2}\mathbb{E}_\nu\|z(\Phi(W))\|_{\mathbb{H}}^2 = \frac{1}{2}\mathbb{E}_\nu\|u(W)\|_{\mathbb{H}}^2 = \frac{1}{2}\mathbb{E}_\mu\|u(X)\|_{\mathbb{H}}^2.$$

Therefore putting pieces together we have

$$\log \mathbb{E}_\mu[e^f] - \varepsilon \leqslant \nu(f) - H(\nu|\mu) = \mathbb{E}_\mu(f(X+I(u(X)))) - \frac{1}{2}\mathbb{E}_\mu\|u(X)\|_{\mathbb{H}}^2.$$

So, for any $\varepsilon>0$ we have found a particular drift u such that

$$\log \mathbb{E}_\mu[e^f] \leqslant \mathbb{E}_\mu(f(X+I(u(X)))) - \frac{1}{2}\mathbb{E}_\mu\|u(X)\|_{\mathbb{H}}^2 + \varepsilon.$$

While if $\log \mathbb{E}_\mu[e^f] = +\infty$ then by the same lemma one has that there exists a sequence of drifts $(u_n)_{n\geqslant 1}$ such that

$$+\infty = \log \mathbb{E}_\mu[e^f] = \sup_n \left[\mathbb{E}_\mu(f(X + I(u_n(X)))) - \frac{1}{2}\mathbb{E}_\mu \|u_n(X)\|_{\mathbb{H}}^2 \right].$$

In both casesn putting together the two inequalities we conclude that

$$\log \mathbb{E}_\mu[e^f] = \sup_u \left[\mathbb{E}_\mu(f(X + I(u(X)))) - \frac{1}{2}\mathbb{E}_\mu \|u(X)\|_{\mathbb{H}}^2 \right]$$

which is our claim. □

Applications to functional analysis

This formula and similar formulas can be used (amazingly) to prove functional inequalities for finite dimensional measures, see for example

- Lehec, Joseph. "Representation Formula for the Entropy and Functional Inequalities." *Annales de l'Institut Henri Poincaré Probabilités et Statistiques* 49, no. 3 (2013): 885–899.
- Lehec, Joseph. . "Short Probabilistic Proof of the Brascamp-Lieb and Barthe Theorems." *Canadian Mathematical Bulletin* 57, no. 3 (September 1, 2014): 585–97. https://doi.org/10.4153/CMB-2013-040-x.
- Borell, Christer. "Diffusion Equations and Geometric Inequalities." *Potential Analysis. An International Journal Devoted to the Interactions between Potential Theory, Probability Theory, Geometry and Functional Analysis* 12, no. 1 (2000): 49–71. https://doi.org/10.1023/A:1008641618547.
- Handel, Ramon van. "The Borell–Ehrhard Game." *Probability Theory and Related Fields* 170, no. 3–4 (April 2018): 555–85. https://doi.org/10.1007/s00440-017-0762-4.

We will not look into these, but they are very interesting.

Applications to probabilitistic problems

Gaussian bounds on functional of Brownian motion.

Theorem 4. *Let (E, d) a metric space and $f: \Omega \to E$ such that there an $e \in E$ for which*

$$d(f(x + I(h)), e) \leqslant c(x)(g(x) + \|h\|_{\mathbb{H}}), \qquad h \in \mathbb{H},$$

for μ-almost every $x \in \Omega$ where $\mu(cg) < \infty$ and $\mu(c^2) < \infty$. Then for all $\lambda > 0$ we have

$$\mathbb{E}_\mu[e^{\lambda d(f(X), e)}] \leqslant e^{\lambda^2 \mu(c^2) + \lambda \mu(cg)}.$$

In particular the r.v. $d(f(X), e)$ has Gaussian tails, i.e.

$$\mathbb{P}_\mu(d(f(X), e) > k) \lesssim C_1 e^{-C_2 k^2}$$

for some $C_1, C_2 > 0$.

Remark 5. Note that if we let $y = x + I(h)$ then $y(t) = x(t) + \int_0^t h(s) \mathrm{d}s$. Note that the natural norm on y is given by the sup norm, i.e.

$$\|y\|_{C([0,1],\mathbb{R}^d)} = \sup_{t \in [0,1]} \left| x(t) + \int_0^t h(s) \mathrm{d}s \right|$$

but on the r.h.s. of the inequality you have to control the L^2 norm of h which corresponds to the H^1 norm of $I(h)$, i.e.

$$\|h\|_{\mathbb{H}} = \|I(h)\|_{\dot{H}^1(\mathbb{R}_+,\mathbb{R}^d)} = \left\|\frac{\mathrm{d}}{\mathrm{d}t} I(h)\right\|_{L^2(\mathbb{R}_+,\mathbb{R}^d)}.$$

This is coherent with the fact that increments of Brownian motion are independent so formally the Wiener measure can be understood as given by

$$\mu(\mathrm{d}\omega) \propto \exp\left(-\frac{1}{2}\int_0^\infty |\dot{\omega}(s)|^2 \mathrm{d}s\right) \mathrm{D}\omega.$$

Proof. By Boué–Dupuis formula and the hypothesis on f

$$\begin{aligned}\log \mathbb{E}_\mu[e^{\lambda d(f(X),e)}] &= \sup_u \mathbb{E}_\mu\left[\lambda d(f(X+I(u)),e) - \frac{1}{2}\|u\|_{\mathbb{H}}^2\right]\\ &\leqslant \sup_u \mathbb{E}_\mu\left[\lambda c(X)(g(X)+\|u\|_{\mathbb{H}}) - \frac{1}{2}\|u\|_{\mathbb{H}}^2\right]\end{aligned}$$

We observe now that the polynomial $\lambda c(X)(g(X)+t) - \frac{1}{2}t^2$ is upperbounded by

$$\lambda c(X) g(X) + \lambda c(X) t - \frac{1}{2}t^2 \leqslant \lambda c(X) g(X) + \frac{1}{2}\lambda^2 c(X)^2 - \frac{1}{2}\underbrace{(t-\lambda c(X))^2}_{\geqslant 0} \leqslant \lambda c(X) g(X) + \frac{1}{2}\lambda^2 c(X)^2$$

therefore

$$\begin{aligned}\log \mathbb{E}_\mu[e^{\lambda d(f(X),e)}] &\leqslant \sup_u \mathbb{E}_\mu\left[\lambda c(X) g(X) + \frac{1}{2}\lambda^2 c(X)^2\right] = \mathbb{E}_\mu\left[\lambda c(X) g(X) + \frac{1}{2}\lambda^2 c(X)^2\right]\\ &= \lambda\mu(cg) + \frac{1}{2}\lambda^2 \mu(c^2).\end{aligned}$$

□

Exercise 1. Take

$$f(x) = \sup_{t,s\in[0,1]} \frac{|x(t)-x(s)|}{|t-s|^\alpha}$$

and prove that is satisfies the hypothesis of the previous theorem. Conclude that

$$\mathbb{E}_\mu\left[\exp\left(\lambda \sup_{t,s\in[0,1]} \frac{|X(t)-X(s)|}{|t-s|^\alpha}\right)\right] \leqslant e^{C_1\lambda^2 + C_2\lambda}$$

for any $\alpha \in (0,1/2)$ any $\lambda > 0$. From this you can also conclude that

$$\mathbb{E}_\mu\left[\exp\left(\rho\left(\sup_{t,s\in[0,1]} \frac{|X(t)-X(s)|}{|t-s|^\alpha}\right)^2\right)\right] < \infty$$

for some $\rho > 0$ small.

Thursday: we continue with applications and with large deviations.