Lecture 25 – 2020.07.16 – 12:15 via Zoom

A brief introduction to rough path theory

(not part of exam)

To know more about it:

Peter K. Friz and Martin Hairer. *A Course on Rough Paths: With an Introduction to Regularity Structures*. Springer, 2014.

Roughly: Rough path theory is a way to make sense of SDEs without using stochastic integrals. (but not only)

Motivation 1. Understanding the Wong–Zakai theorem ('70).

SDE in $\mathbb{R}^d$ with n-dimensional Brownian motion W:

$$\mathrm{d}X_t^i = \sum_{\alpha=1}^n \sigma_\alpha^i(X_t)\mathrm{d}W_t^\alpha, \qquad X_0 = x \in \mathbb{R}^d,$$

Approximate ODE

$$\partial_t X_t^\varepsilon = \sigma(X_t^\varepsilon)\partial_t W_t^\varepsilon, \qquad X_0 = x.$$

with

$$W_t^\varepsilon := \int_{\mathbb{R}} \varepsilon^{-1}\rho((t-s)/\varepsilon)W_s\mathrm{d}s$$

where $\rho: \mathbb{R} \to \mathbb{R}_+$, smooth and with integral one.

Then $W_t^\varepsilon \to W_t$ as $\varepsilon \to 0$ for all t almost surely (and actually almost sure convergence takes place in any Hölder space with index less that $1/2$).

Nonetheless $X^\varepsilon \to Y$ where Y is the process which solves the (different!) SDE

$$\mathrm{d}Y_t^i = \sum_{\alpha=1}^n \sigma_\alpha^i(Y_t)\mathrm{d}W_t^\alpha + C_\rho \sum_{\alpha=1}^n \sum_{j=1}^d \sigma_\alpha^j(Y_t)\nabla^j\sigma_\alpha^i(Y_t)\mathrm{d}t, \qquad t \geqslant 0, i = 1, \ldots, d$$

where here I'm assuming that W takes values in $\mathbb{R}^n$ and Y in $\mathbb{R}^d$ and $\sigma_\alpha: \mathbb{R}^d \to \mathbb{R}^d$ for $\alpha = 1, \ldots, n$ smooth. The constant C_ρ depends on ρ.

Remark that

$$\sum_{\alpha=1}^n \sum_{j=1}^d \sigma_\alpha^j(Y_t)\nabla^j\sigma_\alpha^i(Y_t)\mathrm{d}t = \sum_{\alpha,\beta=1}^n \sum_{j=1}^d \sigma_\alpha^j(Y_t)\nabla^j\sigma_\beta^i(Y_t)\mathrm{d}[W^\alpha, W^\beta]_t$$

(related to the Ito–Stratonovich correction).

Motivation 2. Controlled ODE:

$$\partial_t y(t) = \sigma(y(t))\partial_t x(t), \qquad x(0) = x \in \mathbb{R}^d$$

with $\sigma: \mathbb{R}^d \to L(\mathbb{R}^d; \mathbb{R}^d)$, $x, y \in C^1([0,T], \mathbb{R}^d)$. Consider now a sequence $(x^{(n)})_n \subseteq C^1([0,T], \mathbb{R}^d)$ and $(y^{(n)})_n$ the family of associated solutions to the ODE, i.e.

$$\partial_t y^{(n)}(t) = \sigma(y^{(n)}(t))\partial_t x^{(n)}(t). \qquad y^{(n)}(0) = y_0 \in \mathbb{R}^d$$

No probability here. One can ask what happens if $x^{(n)} \to 0$ (in some topology). Question: what happens to $(y^{(n)})_n$?

One can prove that if $x^{(n)} \to 0$ in $C^\gamma([0,T];\mathbb{R}^d)$ (the γ-Hölder topology) for $\gamma > 1/2$ then $y^{(n)} \to y_0$ (the constant function in y_0) and the convergence holds also in C^γ. Let's assume that σ is smooth. (The optimal regularity for σ depends actually on γ).

However one can also prove that for any $\gamma \in (1/3, 1/2)$ and for *any* $f \in C^{2\gamma}([0,T];A_d)$ where A_d is the space of antisymmetric $d \times d$ matrices. Then I can construct a family $(x^{(n)})_n$ such that $x^{(n)} \to 0$ in C^γ but the soltuions $y^{(n)}$ do not converge to y_0 but to the solution z of the ODE (not quite an ODE)

$$\partial_t z^i(t) = C \sum_{j=1}^d \sigma_\alpha^j(Y_t) \nabla^j \sigma_\beta^i(Y_t) \partial_t f^{\alpha,\beta}(t), \qquad z(0) = y_0.$$

For simplicity you can assume that $f \in C^1([0,T];A_d)$ so that this is really an ODE.

Remark that if $d=1$ there is no space for an interesting case of this result and in $d=1$ is not difficult to prove under sufficient regularity for σ) that $y^{(n)} \to y_0$ (what we expect) (this is related to Doss–Sussmann transformation, that is one can prove that there exists a nice function F such that $y^{(n)}(t) = F(y_0, x^{(n)}(t))$ so uniforml convergence of $(x^{(n)})_n$ is enough to prove uniform convergence $(y^{(n)})_n$.

So the particular phenomenon we see at work here do not really depends on probability, depends however on regularity and on dimension. In particular Hölder regularity $1/2$ has a boundary role, and also $d > 1$ is the interesting situation.

Observation: if $f \in C^\gamma$ with $\gamma > 1/2$ then it has zero quadratic variation.

The core of the problem is that certain non-linear operations (involved in the construction of the solutions to the ODE) are not continuous in C^γ topology if $\gamma < 1/2$.

Model problem is to understand the process of integration, namely the bilinear map

$$I(f,g)(t) = \int_0^t f(s) \mathrm{d}g(s).$$

When $g \in C^1$ this can be understood as

$$I(f,g)(t) = \int_0^t f(s) \partial_s g(s) \mathrm{d}s.$$

More generally one can define this as limit of Riemann sums of the form

$$\sum_i f(\xi_i)(g(\xi_{i+1}) - g(\xi_i))$$

for some given partition $\{\xi_i\}_i$ of $[0,t]$.

We also know that as soon as g is of bounded variation and f is continuous, then the limit exists (Lebesgue–Stiljies integral).

Another situation is that when $f \in C^\gamma$ and $g \in C^\rho$ with (Young regime)

$$\gamma + \rho > 1$$

then the limit exists and is called the Young integral and moreover one has a bilinear and continuous map

$$(f,g) \in C^\gamma \times C^\rho \mapsto I(f,g) \in C^\rho.$$

In particular one can define $\int f(s) \mathrm{d}W(s)$ in a deterministic way where W is BM *provided* the function f is Hölder continuous with exponent $\gamma > 1/2$. Of course this is of NO USE to define SDE because in that setting we really need to take $f(s) = \sigma(X_s)$ and since X has to solve an SDE it must have the same regularity of the BM so we are stuck with $f \in C^\gamma$ for some $\gamma < 1/2$.

Note that if $\gamma = \rho$ then the Young integral is well defined only when $\gamma > 1/2$. This is the key observation to solve controlled ODE in C^γ for $\gamma > 1/2$ (as we did above).

Note also that if I use the same function f (e.g. smooth) then I have

$$I(f,f)(t) = \int_0^t f(s)\mathrm{d}f(s) = \int_0^t f(s)\partial_s f(s)\mathrm{d}s = \frac{1}{2}\int_0^t \partial_s f(s)^2 \mathrm{d}s = \frac{1}{2}(f(t)^2 - f(0)^2).$$

Similarly

$$I(f,g)(t) + I(g,f)(t) = \int_0^t f(s)\partial_s g(s)\mathrm{d}s + \int_0^t g(s)\partial_s f(s)\mathrm{d}s = \frac{1}{2}(f(t)g(t) - f(0)g(0))$$

so this tell us that there is some privileged definition for the symmetric combination $I(f,g)+I(g,f)$, and that this definition make sense for functions f,g or arbitrary regularity. For example if $(f^{(n)},g^{(n)}) \to (0,0)$ as $n \to 0$ (e.g in the uniform topology) then is clear that

$$I(f^{(n)},g^{(n)}) + I(g^{(n)},f^{(n)}) \to 0$$

but is not clear that

$$I(f^{(n)},g^{(n)}) \to 0$$

actually below there is a counterexample.

The condition in the Young integral is optimal in the sense that I can show that the map $(f,g) \mapsto I(f,g)$ is not continuous if $\gamma + \rho < 1$. Take $\alpha > 0$ and

$$f^{(n)}(t) = n^{-\alpha}\sin(nt), \qquad g^{(n)}(t) = n^{-\alpha}\cos(nt),$$

exercise: show that $(f^{(n)},g^{(n)}) \to (0,0)$ in $C^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < \alpha$. Then is a simple matter of computation to show that

$$I(f^{(n)},f^{(n)}) = \frac{1}{2}(f^{(n)}(t)^2 - f^{(n)}(0)^2) \to 0$$

and also

$$I(f^{(n)},g^{(n)}) + I(g^{(n)},f^{(n)}) \to 0$$

but

$$I(f^{(n)},g^{(n)})(t) = \int_0^t n^{-2\alpha}\sin(nt)\mathrm{d}\cos(nt) = n^{1-2\alpha}\underbrace{\int_0^t \sin^2(nt)\mathrm{d}t}_{\to ct>0} \approx ctn^{1-2\alpha}$$

so if $\alpha > 1/2$ this is going to zero but if we take $\alpha = 1/2$ this is converging to a linear function. So let's take $\alpha = 1/2$ then we have that $(f^{(n)},g^{(n)}) \to (0,0)$ in $C^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$ but

$$I(f^{(n)},g^{(n)})(t) \to ct \neq 0 = I(0,0)(t)$$

So in particular the integration map is not continuous in $C^\gamma([0,T];\mathbb{R}) \times C^\gamma([0,T];\mathbb{R})$ for $\gamma < 1/2$.

So when we mix up non-linearity with irregularity, things start getting intersting.

Note that (take $\alpha = 1/2$)

$$I(f^{(n)},f^{(n)})(t) = \int_0^t n^{-1}\sin(nt)\mathrm{d}\sin(nt) = -\int_0^t \sin(nt)\cos(nt)\mathrm{d}t \to 0$$

because the oscillations cancels.

In order to understand the behaviour of the integration in an irregular situation (i.e. outside the Young regime) we can introduce a different way to describe what is the (indefinite) integral of two functions.

I say that $z = I(f, g)$ is the integral of f and g if for all $0 \leqslant s < t$ we have $z(0) = 0$ and

$$z(t) - z(s) = f(s)(g(t) - g(s)) + r(t, s)$$

with $r(t,s)$ *small enough*. In order to determine how small is small enough, the definition has to make sense, meaning, that if z, z' are two functions satisfying the above constraing then I must have $z = z'$. Namely we have

$$z(t) - z(s) = f(s)(g(t) - g(s)) + r(t, s), \qquad z'(t) - z'(s) = f(s)(g(t) - g(s)) + r'(t, s)$$

and taking differences and calling $h = z - z'$, we have that

$$h(t) - h(s) = r(t, s) - r'(t, s)$$

Now observe that if we take $|r(t,s)|, |r'(t,s)| \leqslant C|t-s|^{\zeta}$ for some $\zeta > 1$ (as definition of small enough) then we have that

$$|h(t) - h(s)| \leqslant C|t-s|^{\zeta}$$

so $\partial_t h(t) = 0$ and $h = 0$ so $z = z'$ and the definition is well-posed. So z is the integral of f, g if

$$z(t) - z(s) = f(s)(g(t) - g(s)) + O(|t-s|^{1+}).$$

Read it as: z is the unique function (if it exists) such that

$$|z(t) - z(s) - f(s)(g(t) - g(s))| \lesssim |t-s|^{\zeta}, \qquad 0 \leqslant s < t$$

for some C and $\zeta > 1$.

This is my definition of integral. Any of the above integrals satisfy this definition (classical integral, Young integral) when they are defined.

Take for example B to be a Brownian motion and let's try to define the integral

$$z(t) = \int_0^t \varphi(B(u)) \mathrm{d}B(u)$$

for some nice function φ. This is not possible with Young integral. However we can expand in small time intervals as

$$z(t) - z(s) = \int_s^t \varphi(B(u)) \mathrm{d}B(u) = \varphi(B(s))(B(t) - B(s)) + \int_s^t (\varphi(B(u)) - \varphi(B(s))) \mathrm{d}B(u)$$

Now is clear (by stochastic arguments) that

$$\left| \int_s^t (\varphi(B(u)) - \varphi(B(s))) \mathrm{d}B(u) \right| \lesssim_{\omega} |t-s|^{2\gamma}$$

for any $\gamma \in (1/3, 1/2)$. This is not enough to consider this as a remainder as above. So the idea is that I continue to expand

$$z(t) - z(s) = \varphi(B(s))(B(t) - B(s)) + \underbrace{\int_s^t \int_s^u \nabla\varphi(B(v)) \mathrm{d}B(v) \mathrm{d}B(u)}_{O(|t-s|^{2\gamma})} + \underbrace{\frac{1}{2} \int_s^t \left(\int_s^u \Delta\varphi(B(v)) \mathrm{d}v \right) \mathrm{d}B(u)}_{O(|t-s|^{3\gamma})}$$

The last term can be put into the reminder term and ignored while the second not. One observe also that

$$\int_s^t \int_s^u \nabla\varphi(B(v)) \mathrm{d}B(v) \mathrm{d}B(u) = \nabla\varphi(B(s)) \int_s^t \int_s^u \mathrm{d}B(v) \mathrm{d}B(u) + \underbrace{\int_s^t \int_s^u (\nabla\varphi(B(v)) - \nabla\varphi(B(s))) \mathrm{d}B(v) \mathrm{d}B(u)}_{O(|t-s|^{3\gamma})}$$

But by following the same reasoning above one can show that z is the unique function! such that

$$z(t)-z(s)=\varphi(B(s))(B(t)-B(s))+\nabla\varphi(B(s))\int_s^t\int_s^u \mathrm{d}B(v)\mathrm{d}B(u)+O(|t-s|^{1+}).$$

This description of the integral $\int_s^t \varphi(B(u))\mathrm{d}B(u)$ uses only the knowledge of

$$(B(t))_{t\geqslant 0} \qquad \left(\int_s^t\int_s^u \mathrm{d}B(v)\mathrm{d}B(u)\right)_{t>s>0}.$$

So the integral of a large class of functions is determined by the knowedlge of the path of the Brownian motion and of its *area process*

$$\mathbb{B}^2(s,t):=\int_s^t\int_s^u \mathrm{d}B(v)\mathrm{d}B(u).$$

This is the initial point of the development of rough path theory. The Brownian rough path is the pair

$$(B,\mathbb{B})$$

satisfying

$$\mathbb{B}^2(s,t)=\mathbb{B}^2(s,u)+\mathbb{B}^2(u,t)+(B(t)-B(u))\otimes(B(u)-B(s))$$

and

$$|B(t)-B(s)|+|\mathbb{B}^2(s,t)|^{1/2}\leqslant C|t-s|^{\gamma},$$

for some $\gamma\in(1/3,1/2)$.