Lecture 3 – 28.04.2020 – 12:15 via Zoom

Stochastic differential equations

Existence, uniqueness, various notions thereof, relations between such notions (continued).

Setting. Probability space $(\Omega, \mathscr{F}, \mathbb{P})$, filtration $(\mathscr{F}_t)_{t \geqslant 0}$ right-continuous, completed.

Definition 1. *A **(weak) solution** of the SDE in* $\mathbb{R}^n$

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t \in [0, T]$$

$$X_0 = x \in \mathbb{R}^n$$

is a pair of adapted processes (X, B) *where* $(B_t)_{t \geqslant 0}$ *is a m-dimensional Brownian motion and* b, σ *are coefficients* $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m; \mathbb{R}^n)$ *such that almost surely*

$$\int_0^t |b(X_s)|\mathrm{d}s < \infty, \quad \int_0^t \mathrm{Tr}(\sigma(X_s)\sigma(X_s)^T)\mathrm{d}s < \infty, \qquad t \in [0, T]$$

and that

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s, \qquad t \in [0, T].$$

Note: a weak solution is really the data $(\Omega, \mathbb{P}, (\mathscr{F}_t)_{t \geqslant 0}, X, B)$.

Definition 2. *A **strong solution** to the SDE above is a weak solution such that* X *is adapted to the* $\mathbb{P}$*-completed filtration* $(\mathscr{F}_t^B)_{t \geqslant 0}$ *generated by* B, $\mathscr{F}_t^B := \overline{\sigma(B_s : s \in [0, t])}^{\mathbb{P}}$.

Definition 3. *An SDE has **uniqueness in law** iff two solutions* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t \geqslant 0}, X, B)$, $(\Omega', \mathscr{F}', \mathbb{P}', (\mathscr{F}_t')_{t \geqslant 0}, X', B')$ *are such that*

$$\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{P}'}(X') \in \Pi(C([0, T]; \mathbb{R}^n), \mathscr{B}(C([0, T]; \mathbb{R}^n)))$$

Definition 4. *An SDE has **pathwise uniqueness** if for any two weak solutions* X, X' *defined on the same filt. prob. space and with the **same** BM* B *we have that they are indistinguishable, i.e.*

$$\mathbb{P}(\forall t \in [0, T] : X_t = X_t') = 1.$$

Remark 5. You have to be familiar to the following basic concepts: adapted process, continuous time martingale, local martingale, semimartingale, stochastic integral wrt. semimartingale, (one-)variation of a process, quadratic variation of a processs, co-variation, Riemann-Stiljest integral, Ito formula, Levy caracterisation of Brownian motion (in one dimension).

Theorem 6. *(Cherny) Uniqueness in law implies uniqueness of the law of the pair* (X, B)*, i.e.*

$$\mathrm{Law}_{\mathbb{P}}(X, B) = \mathrm{Law}_{\mathbb{P}'}(X', B').$$

Theorem 7. *(Cherny) Strong existence+uniqness in law* $\Rightarrow$ *pathwise uniqueness.*

Theorem 6 is quite easy to prove if the SDE is one dimensional with $n=m=1$ and $\sigma(x)>0$ everywhere. Indeed observe that if (X,B) is a solution, then the process

$$M_t=\int_0^t \sigma(X_s)\mathrm{d}B_s=X_t-x-\int_0^t b(X_s)\mathrm{d}s \tag{1}$$

is a local martigale and it is measurable wrt. X. But then we have

$$\int_0^t (\sigma(X_s))^{-1}\mathrm{d}M_s=\int_0^t (\sigma(X_s))^{-1}\sigma(X_s)\mathrm{d}B_s=\int_0^t \mathrm{d}B_s=B_t$$

therefore B is X measurable and a consequence $B=\Psi(X)$ and we conclude that

$$\mathrm{Law}_{\mathbb{P}}(X,B)=\mathrm{Law}_{\mathbb{P}}(X,\Psi(X))=\mathrm{Law}_{\mathbb{P}'}(X',\Psi(X'))=\mathrm{Law}_{\mathbb{P}'}(X',B')$$

if X,X' have the same law. Note that $B'=\Psi(X')$ because the map Ψ can be constructed in an almost sure way as follows. From (1) we have that there exists an (adapted) map Φ such that $M_t=\Phi_t(X)$. (and we will have the same for $M'=\Phi(X')$. And remember that for the stochastic integral $\int_0^t (\sigma(X_s))^{-1}\mathrm{d}M_s$ there exists a sequence of (deterministic) partitions $\Pi_n=\{t_1^n,\ldots,t_k^n,\ldots\}$ such that one can express $\int_0^t (\sigma(X_s))^{-1}\mathrm{d}M_s$ as almost sure limit of Riemann sums over the sequence of partitions

$$B_t=\int_0^t (\sigma(X_s))^{-1}\mathrm{d}M_s=\lim_n\sum_k (\sigma(X_{t_k^n}))^{-1}(M_{t_{k+1}^n}-M_{t_k^n})=\lim_n\sum_k (\sigma(X_{t_k^n}))^{-1}(\Phi_{t_{k+1}^n}(X)-\Phi_{t_k^n}(X))=\Psi_t(X)$$

and one can arrange to have the same partition for the primed solution and therefore have $B'=\Psi(X')$ at least $\mathbb{P}'$-a.s. (I skipped the detail of localizing the local martingale M in order to find the deterministic partition).

Let's discuss now the general case. Take $n\geqslant 1$, $m\geqslant 1$ $\sigma:\mathbb{R}^n\to\mathscr{L}(\mathbb{R}^m;\mathbb{R}^n)\approx\mathbb{R}^{n\times m}$.

Let $(\Omega^\sharp,\mathscr{F}^\sharp,\mathbb{P}^\sharp)$ another probability space on which there are two $\mathbb{R}^m$-Brownian motions $W,\bar{W}$. I form the product space $(\tilde{\Omega}=\Omega\times\Omega^\sharp,\tilde{\mathscr{F}}=\mathscr{F}\otimes\mathscr{F}^\sharp,\tilde{\mathbb{P}}=\mathbb{P}\otimes\mathbb{P}^\sharp)$ and on $\tilde{\Omega}$ I consider the solution (X,B) of the SDE together with processes $W,\bar{W}$. Note that $(W,\bar{W})$ is independent of (X,B). Of course $\mathrm{Law}_{\tilde{\mathbb{P}}}(X,B)=\mathrm{Law}_{\mathbb{P}}(X,B)$. For any fixed $x\in\mathbb{R}^n$ consider now $\varphi(x),\psi(x)\in\mathbb{R}^{m\times m}$ such that they are orthogonal projections on orthogonal subspaces:

$$\varphi(x)=\varphi(x)^T,\quad \psi(x)=\psi(x)^T,\quad \psi(x)^2=\psi(x),\quad \varphi(x)^2=\varphi(x),\quad \varphi(x)\psi(x)=0,\quad \varphi(x)+\psi(x)=\mathbb{1}_{n\times n}$$

and such that $\sigma(x)\varphi(x)=\sigma(x)$ and $\sigma(x)\psi(x)=0$. So $\mathrm{Im}(\varphi(x))^\perp=\mathrm{Ker}(\sigma(x))=\mathrm{Im}(\psi(x))$. Now I define two new processes U,V on $\tilde{\Omega}$, with values in $\mathbb{R}^n$ and such that $U_0=V_0=0$ and

$$\mathrm{d}U_t=\varphi(X_t)\mathrm{d}B_t+\psi(X_t)\mathrm{d}W_t$$
$$\mathrm{d}V_t=\psi(X_t)\mathrm{d}B_t+\varphi(X_t)\mathrm{d}\bar{W}_t$$

With this definition we have

$$\mathrm{d}[U^i,U^j]_t=\sum_{k,l}\varphi^{i,k}(X_t)\varphi^{j,l}(X_t)\underbrace{\mathrm{d}[B^k,B^l]_t}_{=\delta_{k,l}t}+\sum_{k,l}\varphi^{i,k}(X_t)\psi^{j,l}(X_t)\underbrace{\mathrm{d}[B^k,W^l]_t}_{=0}$$

$$+\sum_{k,l}\psi^{i,k}(X_t)\varphi^{j,l}(X_t)\underbrace{\mathrm{d}[W^k,B^l]_t}_{=0}+\sum_{k,l}\psi^{i,k}(X_t)\psi^{j,l}(X_t)\underbrace{\mathrm{d}[W^k,W^l]_t}_{=\delta_{k,l}\mathrm{d}t}$$

$$=(\varphi(X_t)\varphi(X_t)^T)^{i,j}\mathrm{d}t+(\psi(X_t)\psi(X_t)^T)^{i,j}\mathrm{d}t=\delta_{i,j}\mathrm{d}t$$

by the properties of φ,ψ. Similarly $\mathrm{d}[V^i,V^j]_t=\delta_{i,j}\mathrm{d}t$ and moreover $\mathrm{d}[U^i,V^j]_t=0$ since $\varphi(x)\psi(x)=0$. We conclude the process (U,V) is a pair of independent $\mathbb{R}^m$-Brownian motions (by the multidimensional version of Levy's caracterisation theorem, we will prove it later on). Now we have

$$B_t=\int_0^t \varphi(X_s)\mathrm{d}U_s,\qquad \int_0^t \sigma(X_s)\mathrm{d}B_s=\int_0^t \sigma(X_s)\varphi(X_s)\mathrm{d}B_s=\int_0^t \sigma(X_s)\mathrm{d}U_s.$$

This implies that $(\tilde{\Omega}, \tilde{\mathbb{P}}, (\tilde{\mathscr{F}}_t^{X,U})_{t\geqslant 0}, X, U)$ is a weak solution to the SDE. I want to prove that V is independent of X. Define the filtration $(\mathscr{G}_t)_{t\geqslant 0}$ given by

$$\mathscr{G}_t = \sigma(U_s, X_s : s \leqslant t) \vee \sigma(V_s : s \geqslant 0).$$

Since U is independent of V, then U is still a $(\mathscr{G}_t)_{t\geqslant 0}$ Brownian motion, which implies in particular that $(U_t)_{t\geqslant 0}$ is independent of $\mathscr{G}_0$ therefore $(\tilde{\Omega}, \tilde{\mathbb{P}}, (\mathscr{G}_t)_{t\geqslant 0}, X, U)$ is still a solution of the SDE.

Now we want to consider the regular conditional probability of $\tilde{\mathbb{P}}$ given $\mathscr{G}_0$ that is the family of probability kernels $\mathbb{Q}: \tilde{\Omega} \to \Pi(\tilde{\Omega})$ such that

$$\mathbb{Q}_\omega(\cdot) = \tilde{\mathbb{P}}(\cdot|\mathscr{G}_0)(\omega), \qquad \text{for } \tilde{\mathbb{P}}\text{-a.e. } \omega \in \tilde{\Omega}.$$

I can do it because I can set up the full theorem in the case where $\tilde{\Omega}$ is the Polish space $\tilde{\Omega} = \mathscr{C}^{n+3m} = C(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$. The probability kernel $\mathbb{Q}$ is unique $\tilde{\mathbb{P}}$-a.s. Observe that $\mathscr{G}_0 = \sigma(V_s : s \geqslant 0)$ since we take a deteministic initial condition for $X_0 = x \in \mathbb{R}^n$.

Observe that almost sure events for $\tilde{\mathbb{P}}$ remains almost sure for $\mathbb{Q}_\omega$ (for $\tilde{\mathbb{P}}$-a.e. $\omega \in \tilde{\Omega}$), i.e.

$$1 = \tilde{\mathbb{P}}(A) \Rightarrow (\mathbb{Q}_\omega(A) = 1, \text{for } \tilde{\mathbb{P}}\text{-a.e. } \omega \in \tilde{\Omega})$$

indeed

$$1 = \tilde{\mathbb{P}}(A) = \int_{\tilde{\Omega}} \mathbb{Q}_\omega(A) \tilde{\mathbb{P}}(\mathrm{d}\omega).$$

By one of the theorems proven in Sheet 0 (this week), we have that $(\tilde{\Omega}, \mathbb{Q}_\omega, (\mathscr{G}_t)_{t\geqslant 0}, X, U)$ is still a weak solution to the SDE for $\tilde{\mathbb{P}}$-a.e. $\omega \in \tilde{\Omega}$. By uniqueness in law of the solutions to the SDE (by assumption), we have that the law under $\mathbb{Q}_\omega$ of X does not depend on ω, i.e.

$$\mathbb{Q}_\omega(X \in \cdot) = \mathrm{Law}_{\mathbb{Q}_\omega}(X) = \mathrm{Law}_{\mathbb{Q}_{\omega'}}(X) \qquad \text{for a.e. } \omega, \omega' \in \tilde{\Omega}.$$

Now

$$\tilde{\mathbb{P}}(X \in A, V \in B) = \int_{\{V \in B\}} \mathbb{Q}_\omega(X \in A) \tilde{\mathbb{P}}(\mathrm{d}\omega) = \int_{\tilde{\Omega}} \mathbb{Q}_{\omega'}(X \in A) \tilde{\mathbb{P}}(\mathrm{d}\omega') \int_{\{V \in B\}} \tilde{\mathbb{P}}(\mathrm{d}\omega) = \tilde{\mathbb{P}}(X \in A) \tilde{\mathbb{P}}(V \in B)$$

We conclude that X, V are independent. Next we are going to prove that $B = B(X, V)$.
