Lecture 4 – 30.04.2020 – 12:15 via Zoom

Stochastic differential equations

Existence, uniqueness, various notions thereof, relations between such notions (continued).

Theorem 1. *(Cherny) If an SDE has uniqueness in law then any weak solution (X,B) has the same law.*

Recall that uniqueness in law means that two weak solutions $(\mathbb{P},X,B)$ and $(\mathbb{P}',X',B')$ satisfy $\mathrm{Law}_{\mathbb{P}}(X)=\mathrm{Law}_{\mathbb{P}'}(X')$.

In order to prove the theorem we constructed a new Polish probability space $\tilde{\Omega}$ which supports the law of a solution (X,B) of the SDE and also two independent BM $W,\bar{W}$. We defined two new $\mathbb{R}^m$-valued BMs with

$$
\begin{aligned}
\mathrm{d}U_t &= \varphi(X_t)\mathrm{d}B_t+\psi(X_t)\mathrm{d}W_t,\\
\mathrm{d}V_t &= \psi(X_t)\mathrm{d}B_t+\varphi(X_t)\mathrm{d}\bar{W}_t.
\end{aligned}
$$

With $\varphi(x)$ the projection on $\ker(\sigma(x))^{\perp}$ and $\psi(x)$ the projection on $\ker(\sigma(x))$. Using uniqueness in law we proved that V is independent of X.

Let us introduce a matrix $\chi(x)\in\mathbb{R}^{m\times n}\approx\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m)$ such that $\chi(x)\sigma(x)=\varphi(x)$ (left inverse to σ) then let

$$
M_t := X_t - x - \int_0^t b(X_s)\mathrm{d}s = \int_0^t \sigma(X_s)\mathrm{d}B_s
$$

then M is local martingale and

$$
\int_0^t \chi(X_s)\mathrm{d}M_s = \int_0^t \chi(X_s)\sigma(X_s)\mathrm{d}B_s = \int_0^t \varphi(X_s)\mathrm{d}B_s = \int_0^t \varphi(X_s)\mathrm{d}U_s
$$

and

$$
B_t = \int_0^t \underbrace{(\varphi(X_s)+\psi(X_s))}_{=1}\mathrm{d}B_s = \int_0^t \varphi(X_s)\mathrm{d}U_s + \int_0^t \psi(X_s)\mathrm{d}V_s = \int_0^t \chi(X_s)\mathrm{d}M_s + \int_0^t \psi(X_s)\mathrm{d}V_s
$$

So we have that B_t can be expressed as a measurable function of (X,V). Therefore there exists a measurable map $\Gamma:\mathscr{C}^n\times\mathscr{C}^m\to\mathscr{C}^m$ such that $B=\Gamma(X,V)$. Therefore $(X,B)=(X,\Gamma(X,V))$. If (X',B') is another weak solution we will have in the same way that $(X',B')=(X',\Gamma(X',V'))$. But X is independent of V, V has the same law of V' (both m-dim BM) and X has the same law of X' (by assumption). So

$$
\mathrm{Law}(X,V)=\mathrm{Law}(X',V')
$$

and as a consequence

$$
\mathrm{Law}(X,B)=\mathrm{Law}(X,\Gamma(X,V))=\mathrm{Law}(X',\Gamma(X',V'))=\mathrm{Law}(X',B').
$$

This concludes the proof of the theorem.

Theorem 2. *(Cherny) Strong existence and uniqueness in law imply path-wise uniqueness.*

Proof. By strong existence there exist a weak solution (X,B) such that $X=\Phi(B)$. By uniqueness in law and the previous theorem we have that any two weak solutions (X,B) and (X',B') have the same law. So now take another weak solution (X',B) on the same probability space of (X,B) and with the same BM. Then we have that

$$
\mathrm{Law}(X',B)=\mathrm{Law}(X,B)=\mathrm{Law}(\Phi(B),B)
$$

It means that $\mathbb{P}(X' = \Phi(B)) = \mathbb{P}(X = \Phi(B)) = 1$ this implies that $\mathbb{P}(X = X') = 1$. So we have pathwise uniqueness. □

Remember that Yamada-Watanabe says that path-wise uniqueness and weak existence implies existence of strong solutions.

Levy's caracterisation of multidimensional BM.

Theorem 3. *Let* $(M_t)_{t\geqslant 0}$ *be a local martingale with values in* $\mathbb{R}^n$ *such that* $M_0 = 0$ *and*

$$[M^i, M^j]_t = \delta_{i,j} t \qquad t \geqslant 0,$$

then $(M_t)_{t\geqslant 0}$ *is a* $\mathbb{R}^n$*-valued Brownian motion.*

Proof. Take $v \in \mathbb{R}^n$ and let $M_t^v = \langle v, M_t \rangle$ a one dimensional local martingale. Note that

$$[M^v]_t = [M^v, M^v]_t = \sum_{i,j} v^i v^j [M^i, M^j]_t = \sum_{i,j} v^i v^j \delta_{i,j} t = \|v\|^2 t$$

Introduce the process

$$\Phi_t^v = \exp\left(i M_t^v + \frac{1}{2}[M_t^v]_t \right) = \exp\left(i M_t^v + \frac{1}{2}\|v\|^2 t \right) = \exp\left(\frac{1}{2}\|v\|^2 t \right) (\cos(M_t^v) + i \sin(M_t^v))$$

observe that

$$|\Phi_t^v| \leqslant \left| \exp\left(i M_t^v + \frac{1}{2}[M_t^v]_t \right) \right| \leqslant \exp\left(\frac{1}{2}\|v\|^2 t \right).$$

So the family $(\Phi_t^v)_{t\in[0,T]}$ is uniformly integrable in any bounded interval $[0, T]$. Moreover by Ito formula

$$\mathrm{d}\Phi_t^v = \exp\left(i M_t^v + \frac{1}{2}[M_t^v]_t \right) \left(i \mathrm{d}M_t^v + \frac{1}{2}\mathrm{d}[M_t^v]_t \right) + \underbrace{\frac{i^2}{2}\exp\left(i M_t^v + \frac{1}{2}[M_t^v]_t \right) \mathrm{d}[M^v]_t}_{\text{Ito correction}}$$

$$= \Phi_t^v \left(i \mathrm{d}M_t^v + \frac{1}{2}\cancel{\mathrm{d}[M_t^v]_t} - \frac{1}{2}\Phi_t^v \cancel{\mathrm{d}[M^v]_t} \right) = i \Phi_t^v \, \mathrm{d}M_t^v,$$

So we have

$$\Phi_t^v = \Phi_0^v + \int_0^t i \Phi_s^v \, \mathrm{d}M_s^v,$$

which shows that $(\Phi_t^v)_{t\in[0,T]}$ is a local martingale (because the stoch. int. $\int_0^t i \Phi_s^v \, \mathrm{d}M_s^v$ is a local mart.) and by the integrability is also a martingale (in that interval). Therefore

$$1 = (\Phi_s^v)^{-1} \Phi_s^v = (\Phi_s^v)^{-1} \mathbb{E}[\Phi_t^v | \mathscr{F}_s] = \mathbb{E}[\Phi_t^v (\Phi_s^v)^{-1} | \mathscr{F}_s]$$

$$= \mathbb{E}\left[\exp\left(i (M_t^v - M_s^v) + \frac{1}{2}([M_t^v]_t - [M_t^v]_s) \right) | \mathscr{F}_s \right]$$

which shows that

$$\mathbb{E}[\exp(i\langle v, M_t - M_s \rangle) | \mathscr{F}_s] = \exp\left(-\frac{\|v\|^2}{2}(t - s) \right)$$

for any $0 \leqslant s \leqslant t \leqslant T$ but since T is arbitrary, the relation is true for any time. First consequence of this relation is that $M_t - M_s$ is independent of $\mathscr{F}_s$ (because the conditional expectation of the complext exponential is non-random. Indeed for any $X \hat{\in} \mathscr{F}_s$ (measurable wrt) one has

$$\mathbb{E}[\exp(i\langle v, M_t - M_s \rangle + i\alpha X)] = \mathbb{E}[\exp(i\langle v, M_t - M_s \rangle)] \mathbb{E}[i\alpha X]$$

(think about it) and by the properties of characteristic functions of vector valued r.v. one has that $M_t - M_s$ is independent of X. Moreover $M_t - M_s$ is a centred Gaussian vector with covariance matrix $\mathbb{1}_{n\times n}(t-s)$.

Using these two facts one prove by induction that for any $0 \leqslant t_1 < t_2 < \cdots < t_n$ we have that $(M_{t_{k+1}} - M_{t_k})_{k=1,\ldots,n-1}$ is an independent family of Gaussian vectors. Since $(M_t)_{t\geqslant 0}$ is continuous and adapted to $(\mathscr{F}_t)_{t\geqslant 0}$ we deduce that $(M_t)_{t\geqslant 0}$ is a n dimensional Brownian motion. □

Some interesting facts come out of it.

Example 4. *(Random rotations)* Let B be a n-dimensional Brownian motion and $(O_t)_{t\geqslant 0}$ be an adapted process made of orthogonal transformations of $\mathbb{R}^n$, i.e. $O_t \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $O_t^T O_t = O_t O_t^T = \mathbb{1}_{n\times n}$. Then consider the $\mathbb{R}^n$ valued local martingale

$$M_t = \int_0^t O_s \mathrm{d}B_s = \left(\sum_{j=1}^n \int_0^t O_s^{i,j} \mathrm{d}B_s^j \right)_{i=1,\ldots,n}, \qquad \mathrm{d}M_t = O_t \mathrm{d}B_t$$

We have

$$[M^i, M^j]_t = \sum_{k,l=1}^n \int_0^t O_s^{i,k} O_s^{j,l} \underbrace{\mathrm{d}[B^k, B^l]_s}_{\delta_{k,l}\mathrm{d}s} = \int_0^t \sum_{k=1}^n O_s^{i,k} O_s^{j,k} \mathrm{d}s = \delta_{i,j} t$$

so by Levy's theorem this process is again a Brownian motion.

Example 5. *(Bessel process)* Let B be n-dimensional Brownian motion starting from $B_0 = x \in \mathbb{R}^n \neq 0$ and consider the process $R_t = |B_t|$ be the Euclidean length of B_t. I want to compute the dynamics of R_t. The function $\varphi(x) = |x|$ is smooth away from the origin and

$$\nabla \varphi(x) = \frac{x}{|x|}, \qquad \nabla^i \nabla^j \varphi(x) = \frac{\delta_{i,j}}{|x|} - \frac{x^i x^j}{|x|^3}, \qquad \mathbb{R}^d \ni x \neq 0.$$

By Ito formula

$$\mathrm{d}R_t = \mathrm{d}\varphi(B_t) = \sum_{i=1}^n \nabla^i \varphi(B_t) \mathrm{d}B_t^i + \frac{1}{2} \sum_{i,j=1}^n \nabla^i \nabla^j \varphi(B_t) \mathrm{d}[B^i, B^j]_t = \underbrace{\sum_{i=1}^n \frac{B_t^i}{|B_t|} \mathrm{d}B_t^i}_{=:\mathrm{d}W_t} + \frac{n-1}{2} \frac{1}{|B_t|} \mathrm{d}t = \mathrm{d}W_t + \frac{n-1}{2} \frac{\mathrm{d}t}{R_t}$$

as least for some small random time interval (in order to be sure that B_t does not touch the origin). Moreover the local martingale $(W_t)_t$ is really a Brownian motion, indeed

$$[W]_t = \int_0^t \sum_{i,j=1}^n \frac{B_s^i}{|B_s|} \frac{B_s^j}{|B_s|} \underbrace{\mathrm{d}[B^i, B^j]_s}_{=\delta_{i,j}\mathrm{d}s} = \int_0^t \mathrm{d}t = t.$$

So (R_t, W_t) is a weak solution of the one dimensional SDE

$$\mathrm{d}R_t = \frac{n-1}{2} \frac{\mathrm{d}t}{R_t} + \mathrm{d}W_t$$

with initial condition $R_0 = |B_0| > 0$. Observe that $R_t > 0$ for any time $t < T_0 = \inf\{t > 0: R_t = 0\}$. Here n has to be integer. But the SDE has a meaning also for $n \in \mathbb{R}$. From the properties of the Brownian motion we know that if $n \geqslant 2$ then $T_0 = +\infty$ almost surely, while if $n = 1$ then $T_0 < \infty$ a.s. What about uniqueness of solutions.

Theorem 6. *For pathwise uniqueness in one dimension see the theorem of Yamada-Watanabe in the Sheet 0, essentially we have pathwise uniqueness as soon as the drift* $b: \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz continuous i.e.*

$$|b(x) - b(y)| \leqslant C|x - y|$$

(same as for ODEs) and the diffusion coefficient $\sigma: \mathbb{R} \to \mathbb{R}$ *is locally* $1/2$*-Hölder continuous, i.e.*

$$|\sigma(x) - \sigma(y)| \leqslant C|x - y|^{1/2}.$$

Theorem 7. *In general dimension pathwise uniqueness holds when both b, σ are locally Lipschitz continuous (sufficient only).*

Therefore the SDE

$$\mathrm{d}R_t = \frac{n-1}{2}\frac{\mathrm{d}t}{R_t} + \mathrm{d}W_t,$$

has pathwise uniqueness away from 0, meaning that given two continuous solutions R, R' with same W and $\mathrm{R}_0 = R'_0 > 0$ and letting

$$T = \inf\{t \geqslant 0 : R_t = 0 \text{ or } R'_t = 0\}$$

then $R_t = R'_t$ for all $t < T$. Indeed in any open set away from 0 the coefficients $\sigma(x) = 1$ and $b(x) = (n-1)/(2x)$ are locally Lipshitz. Which means that the unique strong solution stay positive when $n \geqslant 2$ and that $T_0 = +\infty$ a.s. The process $(R_t)_{t \geqslant 0}$ is called the n-dimensional Bessel process.