Lecture 5 – 2020.05.05 – 12:15 via Zoom

SDE techniques: martingale solutions, time change

Martingale solutions, relations with weak solutions, (uniqueness), time change of martingale solutions, DDS theorem.

Martingale solutions is another technique to caracterise and study solutions of SDEs:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \underbrace{\sigma(X_t)\mathrm{d}B_t}_{\mathrm{d}M_t} \tag{1}$$

In this relation we need to discuss two processes: the solution X and the driving Brownian motion B. We take X to be $\mathbb{R}^n$-valued and B to be $\mathbb{R}^m$-valued and $b: \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m; \mathbb{R}^n)$ measurable and locally bounded.

Martingale solution characterise the process X alone without the need of introducing the driving BM. To start observe the following two facts: if X solve the SDE then

$$M_t = X_t - X_0 - \int_0^t b(X_s)\mathrm{d}s$$

is a local martingale with quadratic variation

$$[M^i, M^j]_t = \int_0^t a^{ij}(X_s)\mathrm{d}s, \qquad i, j = 1, \ldots, n$$

with $a(x) = \sigma(x)\,\sigma(x)^T$ i.e. $a(x)^{i,j} = \sum_{k=1}^m \sigma(x)^{i,k}\sigma(x)^{j,k}$. Similarly for any $f \in C^2(\mathbb{R}^n)$ by Ito formula we have

$$f(X_t) = f(X_0) + \underbrace{\int_0^t \nabla f(X_s) \cdot \sigma(X_s)\mathrm{d}B_s}_{=: M_t^f \text{ (local martingale)}} + \int_0^t \mathscr{L}f(X_s)\mathrm{d}s,$$

with $\mathscr{L}$ a linear operator (*generator*) defined on C^2 functions as

$$\mathscr{L}f(x) = b(x) \cdot \nabla f(x) + \frac{1}{2}\sum_{i,j=1}^n a^{i,j}(x)\nabla_i \nabla_j f(x).$$

Definition 1. *We say that $(X_t)_t$ is martingale solution of the SDE (1) if one of the following equivalent facts holds:*

a) *For any $f \in C^2(\mathbb{R}^n)$ we have that*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)\mathrm{d}s$$

is a local martingale.

b) *The $\mathbb{R}^n$-valued continuous process*

$$M_t = X_t - X_0 - \int_0^t b(X_s)\mathrm{d}s$$

is a local martingale with covariation

$$[M^i, M^j]_t = \int_0^t a^{ij}(X_s)\mathrm{d}s,$$

c) *For any* $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ *we have*

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left[\left(\tfrac{\partial}{\partial s} + \mathscr{L} \right) f \right](s, X_s) \mathrm{d}s$$

is a local martingale.

This formulation is really a description of the law of X. Consider $\mathscr{C} = C(\mathbb{R}_+; \mathbb{R}^n)$ with the Borel σ-field $\mathscr{G}$, the canonical filtration $(\mathscr{G}_t)_{t \geqslant 0}$ and canonical process $Z_t(\omega) = \omega_t$.

Definition 2. *A probability* $\mathbb{P}$ *on* $(\mathscr{C}, \mathscr{G}, (\mathscr{G}_t)_{t \geqslant 0})$ *is a martingale solution (or a solution of the martingale problem) of the SDE if the canonical process* $(Z_t)_t$ *is a martingale solution under* $\mathbb{P}$.

Note that the notion of martingale depends on $\mathbb{P}$.

Theorem 3. *If* $(\mathscr{C}, \mathscr{G}, (\mathscr{G}_t)_{t \geqslant 0}, \mathbb{Q})$ *is a martingale solution to the SDE (1) iff there exist a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *and two processes* (X, B) *over it such that* (X, B) *is a weak solution to the SDE and* $\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{Q}}(Z)$.

Proof. If (X, B) is a weak solution then taking $\mathbb{Q} = \mathrm{Law}_{\mathbb{P}}(X)$ give that $\mathbb{Q}$ is a martingale solution. The more difficult part is to start from a solution of the martingale problem $\mathbb{Q}$ and try to reconstruct a Brownian motion B and then a weak solution. (this reminds us the situation in Cherny's theorem). Indeed if σ is non-degenerate, i.e. there exists a (locally bounded) two side inverse $\sigma(x)^{-1}$ then we could simply take the $\mathbb{Q}$-local martingale

$$M_t = Z_t - Z_0 - \int_0^t b(Z_s) \mathrm{d}s$$

(recall that we are on $\mathscr{C}$ with canonical process Z) and define on $\mathscr{C}$

$$B_t := \int_0^t \sigma(Z_s)^{-1} \mathrm{d}M_s$$

and check that this is indeed a $(\mathbb{Q}, (\mathscr{G}_t)_{t \geqslant 0})$-Brownian motion and that

$$Z_t - Z_0 - \int_0^t b(Z_s) \mathrm{d}s = \int_0^t \mathrm{d}M_s = \int_0^t \sigma(Z_s) \mathrm{d}B_s$$

so (Z, B) is a weak-solution. In this case we can perform the construction on the same probability space. If σ is not invertible we proceed as in Cherny's theorem. We have a left inverse $\chi(x)$ such that $\sigma(x)\chi(x) = \mathbb{1}_{n \times n}$ and $\chi(x)\sigma(x) = \varphi(x)$ where $\varphi(x)$ is the orthogonal projection on $\ker(\varphi(x))^{\perp}$ and we call $\psi(x)$ the orthogonal projection on $\ker(\sigma(x))$. Now we take a larger probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with processes (X, W) such that $\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{Q}}(Z)$ and X, W are independent and W is a m-dimensional Brownian motion. The we set

$$M_t = X_t - X_0 - \int_0^t b(X_s) \mathrm{d}s$$

$$B_t = \int_0^t \chi(X_s) \mathrm{d}M_s + \int_0^t \psi(X_s) \mathrm{d}W_s$$

now is easy to check that $(M_t)_t$ is a local martingale with quadratic variation $\mathrm{d}[M^i, M^j]_t = a^{i,j}(X_t)\mathrm{d}t$ (because under $\mathbb{P}$ the process X is a martingale solution) and moreover $(B_t)_t$ is a Brownian motion.

$$\begin{aligned}
\mathrm{d}[B, B]_t &= \chi(X_s) a(X_s) \chi(X_s)^T \mathrm{d}s + \psi(X_s) \psi(X_s)^T \mathrm{d}s \\
&= \underbrace{\chi(X_s) \sigma(X_s)}_{\varphi(X_s)} \sigma(X_s)^T \chi(X_s)^T \mathrm{d}s + \psi(X_s) \psi(X_s)^T \mathrm{d}s \\
&= (\varphi(X_s) \varphi(X_s)^T + \psi(X_s) \psi(X_s)^T) \mathrm{d}s = \mathbb{1}_{n \times n} \mathrm{d}s.
\end{aligned}$$

□

Remark 4. Note that the notion of martingale solution makes sense also when $\sigma = 0$. Exercise: prove that in this case a process X satisfies the martingale problem iff X is a solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t = b(X_t), \qquad t \geqslant 0.$$

In this case $\mathscr{L} f(x) = b(x) \cdot \nabla f(x)$.

See the book of Rogers and Williams and of Ethier and Kurtz for nice applications of the martingale problem approach.

Mart. prob. were introduced by Stroock and Varadhan (see their book: "Multidimensional diffusion processes")

Remark 5. Uniqueness in law is equivalent to the uniqueness of solutions to the martingale problem.

Remark 6. The notion of martingale problem makes sense even when the process X does not take values in a vector space, e.g. on a manifold $\mathscr{M}$. Indeed note that X solve the martingale problem iff

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathscr{L} f(X_s) \mathrm{d}s$$

is a local (real-valued) martingale for any $f \in C^2(\mathscr{M})$ where

$$\mathscr{L} f = B f + \frac{1}{2} \sum_{\alpha=1}^{m} V_\alpha (V_\alpha f)$$

with $B, (V_\alpha)_{\alpha = 1, \ldots, n}$ are vector fields on $\mathscr{M}$. In the case where $\mathscr{M} = \mathbb{R}^n$ we have

$$B f = b(x) \cdot \nabla f(x) - \frac{1}{2} \sum_{\alpha=1}^{n} (\sigma_\alpha(x) \cdot \nabla \sigma_\alpha(x)) \cdot \nabla f(x), \qquad V_\alpha f = \sigma_\alpha(x) \cdot \nabla f(x)$$

with $(\sigma_\alpha(x))_{\alpha = 1, \ldots, m}$ the rows of the matrix $\sigma(x)$. We will discuss more on detail this application when we are going to study SDE on manifolds and Stratonovich integral.

Time change in martingale problems

Let X be the solution of the SDE (1) in the sense of martingale problem. Let $\varrho(x): \mathbb{R}^n \to \mathbb{R}_+$ which is locally bounded and $\varrho(x) > 0$ everywhere and let

$$A_t := \int_0^t \varrho(X_s) \mathrm{d}s.$$

This process is increasing and continuous assume for simplicity that $A_\infty = +\infty$. Define $T_a = \inf\{t \geqslant 0: A_t \geqslant a\}$ for $a \geqslant 0$. Then $T_0 = 0$ and T is the left inverse to A in the sense that

$$T_{A_t} = \inf\{s \geqslant 0: A_s \geqslant A_t\} = t$$

for all $t > 0$ and is defined for all $a > 0$. Moreover T_a is a stopping time for the filtration generated by $(X_s)_s$. Define the process $Y_a = X_{T_a}$ for all $a > 0$. Now the question is to characterise $(Y_a)_{a \geqslant 0}$. Of course $Y_0 = X_0$. Take $f \in C^2(\mathbb{R}^n)$ and note that

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathscr{L} f(X_s) \mathrm{d}s$$

is a local $(\mathscr{F}_t)_{t\geqslant 0}$-martingale. Therefore the process $(N_a^f)_{a\geqslant 0}$ defined by $N_a^f = M_{T_a}^f$ is a local martingale wrt. the filtraltrion $(\mathscr{G}_a)_{a\geqslant 0}$ defined by $\mathscr{G}_a = \mathscr{F}_{T_a}$ (recall the def of σ-algebra of a stopping time and the optional sampling theorem for continuous martingales with bounded quadratic variation, note also that $T_a \leqslant T_b$ if $a \leqslant b$).

$$N_a^f = M_{T_a}^f = f(X_{T_a}) - f(X_0) - \int_0^{T_a} \mathscr{L}f(X_s)\mathrm{d}s = f(Y_a) - f(Y_0) - \int_0^{T_a} \mathscr{L}f(X_s)\mathrm{d}s.$$

To conclude observe that by doing the change of variables $s = s(b)$ such that $b = A_s$ or $T_b = s$

$$\mathrm{d}b = \varrho(X_s)\mathrm{d}s$$

therefore

$$\mathrm{d}s = \frac{\mathrm{d}b}{\varrho(X_s)}$$

and

$$\int_0^{T_a} \mathscr{L}f(X_s)\mathrm{d}s = \int_0^a \mathscr{L}f(Y_b)\mathrm{d}s(b) = \int_0^a \mathscr{L}f(Y_b)\frac{\mathrm{d}b}{\varrho(X_s)} = \int_0^a \mathscr{L}^\varrho f(Y_b)\mathrm{d}b$$

with

$$\mathscr{L}^\varrho f(x) = \frac{1}{\varrho(x)}\mathscr{L}f(x) = \underbrace{\frac{1}{\varrho(x)}b(x)}_{b^\varrho} \cdot \nabla f(x) + \frac{1}{2}\sum_{i,j=1}^n \underbrace{\frac{1}{\varrho(x)}a^{i,j}(x)}_{=\sigma^\varrho(\sigma^\varrho)^T} \nabla_i \nabla_j f(x).$$

So $(Y_a)_{a\geqslant 0}$ solves the martingale problem associated to the generator $\mathscr{L}^\varrho$ namely, is associated to the SDE

$$\mathrm{d}Z_a = b^\varrho(Z_a)\mathrm{d}a + \sigma^\varrho(Z_a)\mathrm{d}B_a, \qquad a \geqslant 0$$

where $b^\varrho(x) = b(x)/\varrho(x)$ and $\sigma^\varrho(x) = \sigma(x)/(\varrho(x))^{1/2}$.
