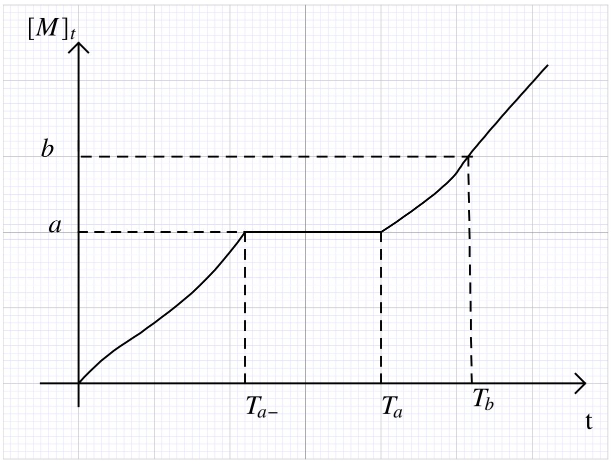
Lecture 6 – 2020.05.07 – 12:15 via Zoom

SDE techniques: martingale solutions, time change

Martingale solutions, relations with weak solutions, (uniqueness), time change of martingale solutions, DDS Brownian motion, one dimensional diffusions.

Dambis, Dubins–Schwarz theorem says that "any continuous local martingale" is the time-change of a Brownian motion. Let us see how.

Let $(M_t)_{t\geqslant 0}$ cont. loc. mart. and consider its quadratic variation $([M]_t)_{t\geqslant 0}$, this is a continuous non-decreasing process. I'm looking for a time-change which "straighten out" the quadratic variation of M.



We define

$$T_a = \inf\{t\geqslant 0: [M]_t > a\}$$

which is the right inverse of $[M]$, in the sense that

$$[M]_{T_a} = a$$

by continuity of $[M]$. However is only right-continuous, and if $t\in[T_{a-},T_a]$ then $M_t = a$. $T_{a-} := \lim_{\varepsilon\downarrow 0} T_{a-\varepsilon}$.

In order for T_a to be a stopping time we need that $(\mathscr{F}_t)_{t\geqslant 0}$ is right continuous, this will be one of our assumptions. We also need that $[M]_\infty = +\infty$ (this assumption can be removed).

We use now $a\mapsto T_a$ as time change. Define a new filtration $(\mathscr{G}_a)_{a\geqslant 0}$ as $\mathscr{G}_a = \mathscr{F}_{T_a}$.

Now let

$$B_a := M_{T_a},$$

We want to prove that B is a Brownian motion via Levy's characterisation.

Let's prove continuity. It is clear that B_a is right-continuous since M is continuous and $T_\cdot$ is right-continuous. However $B_{a-} = \lim_{\varepsilon\downarrow 0} B_{a-\varepsilon} = M_{T_{a-}}$ and one observe that

$$\mathbb{P}(\exists a: M_{T_{a-}} \neq M_{T_a}) = 0.$$

The point here is that if $U < V$ are two stopping times and $[M]_U = [M]_V$ then $M_U = M_V$ a.s. Now we have $[M]_{T_{a-}} = a = [M]_{T_a}$ and $T_{a-} < T_a$ are stopping times, so

$$B_{a-} = M_{T_{a-}} = M_{T_a} = B_a$$

a.s. So $(B_a)_{a\geqslant 0}$ is continuous.

Let's prove that is a local martingale. We have $B_a = M_{T_a} = M_\infty^{T_a}$ recall that $M_t^T = M_{T\wedge t}$ is the stopped process. We have $[M^{T_a}]_\infty = [M]_{T_a} = a$ so $\mathbb{E}[B_a^2] = \mathbb{E}[[M^{T_a}]_\infty] = a$. By optional stopping we have for any $a > b$

$$\mathbb{E}[B_a|\mathscr{G}_b] = \mathbb{E}[M_\infty^{T_a}|\mathscr{F}_{T_b}] = M_{T_b} = B_b \qquad a.s.$$

so $(B_a)_{a\geqslant 0}$ is a local martingale.

We need now to compute its quadratic variation.

$$\mathbb{E}[B_a^2 - B_b^2|\mathscr{G}_b] = \mathbb{E}[(M_\infty^{T_a})^2 - (M_\infty^{T_b})^2|\mathscr{F}_{T_b}] = [M]_{T_a} - [M]_{T_b} = a - b$$

so the process $B_a^2 - a$ is a local martigale therefore $\langle B\rangle_a = a = [B]_a$.

And by Levy's theorem we conclude that $(B_a)_{a\geqslant 0}$ is a Brownian motion. Moreover

$$B_{[M]_t} = M_{T_{[M]_t}} = M_t \qquad a.s.$$

indeed $T_{[M]_t} \geqslant t$ without equality in general, but $[M]_{T_{[M]_t}} = [M]_t$ so $M_{T_{[M]_t}} = M_t$.

Theorem 1. *(Dambis, Dubins–Schwarz Brownian motion) Let $(M)_{t\geqslant 0}$ be a continuous local martingale for a right-continuous filtration $(\mathscr{F}_t)_{t\geqslant 0}$ then there exists a Brownian motion $(B_a)_{a\geqslant 0}$ (maybe in an enlarged probability space) such that $M_t = B_{[M]_t}$.*

Remark 2. If $[M]_\infty = +\infty$ then there is no need to enlarge the probability space.

Example 3. A first application: if $M_t = \int_0^t G_s \mathrm{d}W_s$ where W is a BM. Then $[M]_t = \int_0^t G_s^2 \mathrm{d}s$ and therefore

$$M_t = B_{\int_0^t G_s^2 \mathrm{d}s}$$

where B is the DDS BM of M.

Historically this has some interest because in the '40 Dœblin discovered Ito formula independently of Ito (but this was not known until 2000 because his results were in a letter which was sealed and to be opened not before that date). In Dœblin's approach there were no stochastic calculus, at the placve of stochastic integrals he was using time changed Brownian motions. His approach was limited to one dimension. For example, if W is a BM and $f \in C^2$ then

$$f(W_t) = f(W_0) + \frac{1}{2}\int_0^t f''(W_s)\mathrm{d}s + B_{\int_0^t f'(W_s)^2 \mathrm{d}s}$$

for some BM B.

If $(G_s)_{s\geqslant 0}$ is a deterministic function then this shows that $\left(\int_0^t G_s \mathrm{d}W_s\right)_{t\geqslant 0}$ is a Gaussian process.

Example 4. (Path-wise properties of martingales) Let B be the DDS BM of a local martingale M then we know that for any $\gamma < 1/2$ there exists a random constant C such that almost surely $C < \infty$ and

$$|B_a - B_b| \leqslant C|b-a|^\gamma, \qquad 0 \leqslant a, b \leqslant 1.$$

Therefore

$$|M_t - M_s| = |B_{[M]_t} - B_{[M]_s}| \leqslant C|[M]_t - [M]_s|^\gamma, \qquad 0 \leqslant t, s \leqslant T.$$

almost surely on the event $\{[M]_T \leqslant 1\}$.

Time-change of SDEs

Take the SDE (Y, W) where W is a BM and

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t,$$

where $\sigma(x) > 0$ for all $x \in \mathbb{R}$. Using the DDS BM B of Y we have

$$Y_t = B_{A_t}, \qquad A_t := [Y]_t = \int_0^t \sigma(Y_s)^2 \mathrm{d}s,$$

Let $(T_a)_{a \geqslant 0}$ be the inverse of $(A_t)_{t \geqslant 0}$, i.e. $A_{T_a} = a$ and $T_{A_t} = t$ (inverse exist since $(A_t)_{t \geqslant 0}$ is strictly increasing)

$$\mathrm{d}A_t = \sigma(Y_t)^2 \mathrm{d}t, \qquad \mathrm{d}t = \frac{\mathrm{d}A_t}{\sigma(Y_t)^2}$$

$$T_a = \int_0^a \frac{\mathrm{d}b}{\sigma(Y_{T_b})^2} = \int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2}$$

since $Y_{T_a} = B_a$. So aposteriori if B is a Brownian motion and I define

$$T_a = \int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2},$$

and I take A to be the inverse of T then $Y_t := B_{A_t}$ satisfies the SDE

$$\mathrm{d}Y_t = \sigma(Y_t) \mathrm{d}W_t$$

with some BM W. (This could be obtained also via the time-change in the martingale problem). Note that the time-change can be reconstructed back only knowing the trajectory of the BM. This can be used to find solutions to SDE.

Example 5. (Degenerate diffusion coefficient) Consider $\sigma(x) = |x|^\alpha$ for $x \in \mathbb{R}$ and $\alpha \in (0, 1/2)$. Let $(B_a)_{a \geqslant 0}$ be a 1d Brownian motion starting at $y \in \mathbb{R}$ and define

$$T_a := \int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2}.$$

In order to be sure that $(T_a)_{a \geqslant 0}$ is well defined we need only to be sure that $T_a < \infty$ a.s. Note that

$$\mathbb{E}[T_a] = \mathbb{E}\left[\int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2}\right] = \int_0^a \mathbb{E}\left[\frac{1}{|B_b|^{2\alpha}}\right] \mathrm{d}b = \int_0^a \left[\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{2b}}}{(2\pi b)^{1/2}} \frac{1}{|x|^{2\alpha}} \mathrm{d}x\right] \mathrm{d}b < \infty$$

by Fubini and using that $2\alpha < 1$ to have integrability near $x = 0$. We can define $t \mapsto A_t(\omega)$ to be the inverse of $a \mapsto T_a(\omega)$ (pathwise) and we can define $Y_t = B_{A_t}$. By our previous discussion this is a solution to the SDE

$$\mathrm{d}Y_t = \sigma(Y_t) \mathrm{d}W_t = |Y_t|^\alpha \mathrm{d}W_t$$

for some BM W. So we proved that this SDE has a non-trivial weak solution starting from any $Y_0 = y \in \mathbb{R}$, but now note that this SDE has also the solution $Y_t = 0$ if $y = 0$. So there is no uniqueness in law for this SDE. We have also pathwise non-uniqueness since the two weak solutions (Y, W) and $(0, W)$ do not coincide.

So in general we cannot expect uniqueness in law in one dimension if the diffusion coefficient is degenerate, i.e. $\sigma(x) = 0$ for some x.

Note that by the Yamada–Watanabe (existence) theorem as soon as $\sigma(x) = |x|^\alpha$ with $\alpha \geqslant 1/2$ we have pathwise uniqueness.

Next week: some remarks on one dimensional diffusions. Girsanov's theorem. Doob's transform.