Lecture 7 – 2020.05.12 – 12:15 via Zoom

SDE techniques: martingale solutions, time change

Uniqueness of martingale solutions, one dimensional diffusions.

1 Uniqueness of the martingale problem for a diffusion

$\mathscr{C} = \mathscr{C}^n = C(R_+, \mathbb{R}^n)$ with its Borel σ-algebra $\mathscr{F}$ and canonical process $(X_t)_{t\geqslant 0}$ with associated filtration $(\mathscr{F}_t)_{t\geqslant 0}$. Remember that with $\Pi(\mathscr{C})$ we denote the probability measures on the path space $\mathscr{C}$.

Consider the generator $\mathscr{L}$ defined for any $f \in C^2(\mathbb{R}^n)$ as

$$\mathscr{L}f(x) = b(x) \cdot \nabla f(x) + \frac{1}{2}\mathrm{Tr}(a\nabla^2 f(x)), \qquad x \in \mathbb{R}^n,$$

with measurable and bounded coefficients.

Definition 1. *We say that $\mathbb{P}$ on $(\mathscr{C}, (\mathscr{F}_t)_{t\geqslant 0})$ is a solution of the martingale problem for the generator $\mathscr{L}$ iff for any $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$*

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + \mathscr{L}) f(s, X_s)\mathrm{d}s \qquad (1)$$

is a $\mathbb{P}$-martingale wrt. $(\mathscr{F}_t)_{t\geqslant 0}$.

We want to discuss the uniqueness of such solutions, meaning the following.

Definition 2. *We say that the martingale problem (1) has unique solution if any two solutions $\mathbb{P}, \mathbb{Q} \in \Pi(\mathscr{C})$ of the martingale problem such that $\mathrm{Law}_{\mathbb{P}}(X_0) = \mathrm{Law}_{\mathbb{Q}}(X_0)$ then $\mathbb{P} = \mathbb{Q}$.*

This notion corresponds directly with the uniqueness in law of the corresponding weak solutions. It is enough that $\mathbb{P}, \mathbb{Q}$ coincide on finite dimensional distributions.

Let use observe that if $\varphi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ is a solution to the (parabolic) PDE (Kolmogorov backward equation)

$$\partial_t \varphi(t, x) = \mathscr{L}\varphi(t, x), \qquad t \geqslant 0, x \in \mathbb{R}^n, \qquad (2)$$

Note that $(\partial_s + \mathscr{L})\varphi(r - s, X_s) = 0$ for any $r > s$, therefore for any $r > 0$ and any $t \in [0, r]$ the process

$$M_t^r = \varphi(r - t, X_t) - \varphi(r, X_0) - \int_0^t (\partial_s + \mathscr{L})\varphi(r - s, X_s)\mathrm{d}s = \varphi(r - t, X_t) - \varphi(r, X_0)$$

is a martingale under any solution $\mathbb{P}$ of the martingale problem associated to $\mathbb{P}$. Now $M_r^r = \varphi(0, X_r) - \varphi(r, X_0)$ so

$$0 = \mathbb{E}_{\mathbb{P}}[M_r^r - M_t^r | \mathscr{F}_t] = \mathbb{E}_{\mathbb{P}}[\varphi(0, X_r) - \varphi(r - t, X_t) | \mathscr{F}_t]$$

tells me that for any $r \geqslant t$ we have

$$\mathbb{E}_{\mathbb{P}}[\varphi(0, X_r) | \mathscr{F}_t] = \mathbb{E}_{\mathbb{P}}[\varphi(r - t, X_t) | \mathscr{F}_t] = \varphi(r - t, X_t), \qquad \mathbb{P} - a.s.$$

So the value of this expectation essentially do not depends on which solution of the martingale problem we get

$$\mathbb{E}_{\mathbb{P}}[\varphi(0, X_r)] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\varphi(0, X_r) | \mathscr{F}_0]] = \mathbb{E}_{\mathbb{P}}[\varphi(r, X_0)]$$

and if $\mathbb{Q}$ is another solution with $\text{Law}_{\mathbb{Q}}(X_0) = \text{Law}_{\mathbb{P}}(X_0)$ then we conclude that

$$\mathbb{E}_{\mathbb{P}}[\varphi(0, X_r)] = \mathbb{E}_{\mathbb{Q}}[\varphi(0, X_r)]$$

for any $r \geqslant 0$. Let us assume know that the Kolmogorov backward equation has solution for any initial condition $\psi \in C_0^\infty(\mathbb{R}^n)$ (where the 0 means compactly supported). This implies that if we use such solutions in the argument above we get that for any $\psi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\mathbb{E}_{\mathbb{P}}[\psi(X_r)] = \mathbb{E}_{\mathbb{Q}}[\psi(X_r)]$$

and this implies that

$$\text{Law}_{\mathbb{P}}(X_r) = \text{Law}_{\mathbb{Q}}(X_r) \tag{3}$$

for any $r \geqslant 0$. So we deduced that the one time marginals of $\mathbb{P}$ and $\mathbb{Q}$ coincide. Now let $\psi \in C_0^\infty(\mathbb{R}^n)$ and let φ^ψ to be the solution of (2) such that $\varphi(0, x) = \psi(x)$ for all $x \in \mathbb{R}^n$ then as we already seen $\mathbb{E}_{\mathbb{P}}[\psi(X_r)|\mathscr{F}_t] = \varphi^\psi(r - t, X_t)$, therefore for any $r_1 > r_2 \geqslant 0$ we have for any bounded and measurable $g: \mathbb{R}^n \to \mathbb{R}^n$

$$\mathbb{E}_{\mathbb{P}}[\psi(X_{r_1}) g(X_{r_2})] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\psi(X_{r_1})|\mathscr{F}_{r_2}] g(X_{r_2})] = \mathbb{E}_{\mathbb{P}}[\underbrace{\varphi^\psi(r_1 - r_2, X_{r_2}) g(X_{r_2})}_{\tilde{g}(X_{r_2})}]$$

$$\underset{\text{eq.(3)}}{=} \mathbb{E}_{\mathbb{Q}}[\varphi^\psi(r_1 - r_2, X_{r_2}) g(X_{r_2})] = \mathbb{E}_{\mathbb{Q}}[\psi(X_{r_1}) g(X_{r_2})]$$

since ψ and g are arbitrary we conclude that

$$\text{Law}_{\mathbb{P}}(X_{r_1}, X_{r_2}) = \text{Law}_{\mathbb{Q}}(X_{r_1}, X_{r_2}).$$

We can continue by induction and deduce that $\mathbb{P}, \mathbb{Q}$ have the same finite dimensional marginals, and therefore are equal as probability measures on $\mathscr{C}$. (think about it). Moreover note that we also have for any $r > t$

$$\mathbb{E}_{\mathbb{P}}[\psi(X_r)|\mathscr{F}_t] = \varphi^\psi(r - t, X_t),$$

which implies that the process $(X_t)_{t \geqslant 0}$ under $\mathbb{P}$ is a Markov process, indeed for any $t_1 < \cdots < t_n < r$ we have

$$\mathbb{E}_{\mathbb{P}}[\psi(X_r) g(X_{t_1}, \ldots, X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}[\psi(X_r)|\mathscr{F}_{t_n}] g(X_{t_1}, \ldots, X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\varphi^\psi(r - t_n, X_{t_n}) g(X_{t_1}, \ldots, X_{t_n})]$$

but also

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}[\psi(X_r)|X_{t_n}] g(X_{t_1}, \ldots, X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\varphi^\psi(r - t_n, X_{t_n}) g(X_{t_1}, \ldots, X_{t_n})]$$

from which we get

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}[\psi(X_r)|X_{t_n}] g(X_{t_1}, \ldots, X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\psi(X_r) g(X_{t_1}, \ldots, X_{t_n})]$$

and by a monotone class argument one deduce that

$$\mathbb{E}[\psi(X_r)|X_{t_n}] = \mathbb{E}[\mathbb{E}[\psi(X_r)|X_{t_n}]|\mathscr{F}_{t_n}] = \mathbb{E}[\psi(X_r)|\mathscr{F}_{t_n}]$$

for any $\psi \in C_0^\infty(\mathbb{R}^n)$ which approximates any continuous function and then also indicator functions of open sets from which we conclude that it is true for any ψ which is bounded and measurable. This proves the Markov property of $(X_t)_{t \geqslant 0}$ under $\mathbb{P}$.

Theorem 3. *Assume that the Kolmogorov backward PDE*

$$\partial_t \varphi(t, x) = \mathscr{L} \varphi(t, x), \qquad \varphi(0, \cdot) = \psi$$

has a solution $\varphi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ for any $\psi \in C_0^\infty(\mathbb{R}^n)$ then the martingale problem associated to $\mathscr{L}$ in the sense of Definition 1 has a unique solution in the sense of Definition 2. (and as a consequence uniqueness of weak solutions to the associated SDE).

Remark 4. This reduces the uniqueness problem to a problem about *existence* of enough regular solutions to a PDE. Note that the set of initial conditions $C_0^\infty(\mathbb{R}^n)$ could be replaced by any set $\mathscr{D}$ with the property that if two probability measures $\mu, \nu \in \Pi(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^n} f(x)\nu(\mathrm{d}x), \qquad f \in \mathscr{D}$$

then $\mu = n$, i.e. $\mathscr{D}$ is a determining (or separating) class for $\Pi(\mathbb{R}^n)$.

Remark 5. What about existence of solutions to the martingale problem.

a) (Construction of the weak solution SDE) maybe strong solutions via fixpoint arguments, or time-change, or Girsanov transformation (to be seen), Doob's transform.

b) (Compactness arguments) Assume that we have a sequence of probabilities $(\mathbb{P}^n)_n$ on $\mathscr{C}$ such that $\mathbb{P}^n$ solve the martingale problem wrt. $\mathscr{L}^n$ (some generator). Assume also that we can show pointwise convergence of $\mathscr{L}^n$ to a limiting generator $\mathscr{L}$, in the sense that for any f "in a large class of functions" we have that $\mathscr{L}^n f(x) = \mathscr{L} f(x)$ uniformly in $x \in \mathbb{R}^n$. Assume also that the family $(\mathbb{P}^n)_n$ is tight on $\mathscr{C}$, then one can show that any accumulation point of $(\mathbb{P}^n)_n$ wrt. to the weak topology of probability measures is a solution of the martingale problem for $\mathscr{L}$.

c) (Markov process theory) If one can construct the semigroup $(P_t)_{t\geqslant 0}$ in the space of continuous functions $C(\mathbb{R}^n)$, associated to the operator $\mathscr{L}$ in the sense that $\partial_t P_t = \mathscr{L} P_t$ in the sense of Hille–Yoshida theory. Then one can construct a measure $\mathbb{P}$ using P to specify the finite dimensional distributions and then prove that it is a solution of the martingale problem. (this is stated here very vaguely).

Theorem 6. *(Stroock–Varadhan) Assume b, σ is are bounded measurable functions and a is bounded from below away from zero (in the sense of symmetric matrices) then there exists a solution to the martingale problem for $\mathscr{L}$ and the martingale problem for $\mathscr{L}$ has a unique solution.*

The condition on a means that there exists $\lambda > 0$ such that $\langle v, a(x)v\rangle_{\mathbb{R}^n} \geqslant \lambda \|v\|_{\mathbb{R}^n}^2$ for any $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ (ellipticity condition).

There is no further regularity requirement on the coefficients, i.e. they can be discontinuous.