Lecture 8 – 2020.05.14 – 12:15 via Zoom

SDE techniques: one dimensional diffusions, Girsanov

Uniqueness of martingale solutions, one dimensional diffusions.

Theorem 1. *(Skhorohod) Assume b,σ is are bounded measurable functions then there exists a weak solution of the SDE (equivalently, a solution of the martingale problem for $\mathscr{L}$).*

Theorem 2. *(Stroock–Varadhan) Assume b is a bounded measurable function, σ is continuous and $a=\sigma\sigma^T$ is bounded from below away from zero (in the sense of symmetric matrices) then the martingale problem for $\mathscr{L}$ has a unique solution.*

Remark 3. Most part of the theory exposed so far (e.g. pathwise uniqueness under Lipshitz conditions, Yamada–Watanabe theorem, Cherny's theorem, characterisation of martingale solutions/weak SDE) hold under the more general assumption that the coefficients of the SDE b,σ are adapted function of the “full history” of the process X, in the sense that

$$b:\mathbb{R}_+\times\mathscr{C}^n\to\mathbb{R}^n,\qquad \sigma:\mathbb{R}_+\times\mathscr{C}^n\to\mathscr{L}(\mathbb{R}^m,\mathbb{R}^n),$$

such that the processes $(b(t,X))_{t\geqslant 0}$, $(\sigma(t,X))_{t\geqslant 0}$ are adapted to the filtration generated by X. Of course the Lipschitz condition on b,σ has to be read in the sense of the Banach space $C([0,t];\mathbb{R}^n)$, i.e.

$$|b(t,f)-b(t,g)|+|\sigma(t,f)-\sigma(t,g)|\leqslant C_{b,\sigma}\|f-g\|_{C([0,t];\mathbb{R}^n)}$$

for all $t\geqslant 0$ and all $f,g\in\mathscr{C}^n=C(\mathbb{R}_+;\mathbb{R}^n)$. However the solutions of the SDE are not anymore in general Markov processes. Sometimes the SDE is called Markovian if the coefficients depends only on the current state, i.e. $b(t,f)=b(t,f(t))$, $\sigma(t,f)=\sigma(t,f(t))$.

---

1 One dimensional (Markovian) diffusions

Let X be the solution of the SDE on the interval $(\alpha,\beta)\subset\mathbb{R}$

$$\mathrm{d}X_t=b(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}B_t$$

where $b:(\alpha,\beta)\to\mathbb{R}$ and $\sigma:(\alpha,\beta)\to\mathbb{R}$ are continuous functions such that $\sigma(x)>0$ for any $x\in(\alpha,\beta)$. The combination of time-change and space transformation allows a quite complete description of such kind of SDE.

Let's assume that $X_0=x\in(\alpha,\beta)$ and that

$$\tau=\inf\{t\geqslant 0:X_t\notin(\alpha,\beta)\}=\sup_n\inf\{t\geqslant 0:X_t\in(\alpha+1/n,\beta-1/n)^c\}$$

the exit time of X from (α,β).

Coordinate transformation. Take a function $\varphi\in C^2((\alpha,\beta);\mathbb{R})$ and let $Y_t=\varphi(X_t)$. By appling Ito formula we have

$$\mathrm{d}Y_t=\sigma(X_t)\varphi'(X_t)\mathrm{d}B_t+\left(b(X_t)\varphi'(X_t)+\frac{1}{2}\sigma(X_t)^2\varphi''(X_t)\right)\mathrm{d}t,\qquad t\in[0,\tau].$$

Assume that $\varphi'(x) > 0$ so that φ si bijective onto its image and let φ^{-1} its inverse, then $X_t = \varphi^{-1}(Y_t)$ and if moreover φ satisfies

$$b(x)\varphi'(x) + \frac{1}{2}\sigma(x)^2\varphi''(x) = 0, \qquad x \in (\alpha, \beta) \tag{1}$$

then Y solves the SDE

$$\mathrm{d}Y_t = \tilde{\sigma}(Y_t)\mathrm{d}B_t, \qquad t \in [0, \tau],$$

with $\tilde{\sigma}(y) = (\sigma\varphi') \circ \varphi^{-1}(y) > 0$ and

$$\tau = \inf\{t \geqslant 0 : X_t \notin (\alpha, \beta)\} = \inf\{t \geqslant 0 : Y_t \notin (\varphi(\alpha), \varphi(\beta))\}.$$

Time change. Then Y is a local martingale (up to time τ) and if we let $A_t = [Y]_t = \int_0^t \tilde{\sigma}(Y_s)^2 \mathrm{d}s$ its quadratic variation, then we can define the time changed process $Z_a = Y_{T_a}$ with $T = A^{-1}$. We know that $(Z_a)_{a \geqslant 0}$ (maybe on a larger probability space) is a Brownian motion up to time σ given by $\sigma = A_\tau$ which corresponds to

$$\sigma = \inf\{a \geqslant 0 : Z_a \notin (\varphi(\alpha), \varphi(\beta))\}$$

and that

$$Y_t = Z_{A_t}, \qquad T_a = \int_0^a \frac{\mathrm{d}b}{\tilde{\sigma}(Z_b)^2}, \qquad t \in [0, \tau], \quad a \in [0, \sigma].$$

So overall we can say that by "streching" space and time, any one dimensional diffusion is nothing more than a Brownian motion.

Exercise 1. Perform the coordinate transformation and the change of time on the martingale problem to arrive to the same conclusion.

On the other hand we can go back, i.e. start from the Brownian motion $(Z_a)_{a \geqslant 0}$ perform the time change T_a given above to obtain $Y_t = Z_{A_t}$ and then perform the coordinate transformation in the backward direction to obtain that $X_t = \varphi^{-1}(Y_t) = \varphi^{-1}(Z_{A_t})$ is the solution of the original SDE. So the original SDE has uniqueness in law because we have been able to express its law as a measurable transformation of the law of the Brownian motion. That is $X^\tau = \Phi(Z^\sigma)$ which implies $\mathrm{Law}(X^\tau) = \Phi_*(\mathrm{Law}(Z^\sigma))$.

Theorem 4. *Any solution X of any one dimensional SDE on (α, β) with $\sigma: (\alpha, \beta) \to \mathbb{R}_{>0}$ has the from $X_t = \varphi^{-1}(Z_{A_t})$ for $t \in [0, \tau]$ where τ is the exist time of X from (α, β), φ is the unique (up to shift and rescaling) solution of the ODE (1) such that $\varphi'(x) > 0$ for all $x \in (\alpha, \beta)$ and $A = T^{-1}$ with*

$$T_a = \int_0^a \frac{\mathrm{d}b}{\tilde{\sigma}(Z_b)^2}.$$

In particular such SDE has uniqueness in law.

Let us justify the fact that the ODE (1) has a unique solution with positive derivative (up to shift and rescaling). Note that φ has to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi'(x) = -2\frac{b(x)}{\sigma(x)^2}\varphi'(x), \qquad x \in (\alpha, \beta).$$

This is an ODE for $\varphi'(x)$ which is solved by

$$\varphi'(x) = B\exp\left(-2\int_{x_0}^x \frac{b(z)}{\sigma(z)^2}\mathrm{d}z\right), \qquad x \in (\alpha, \beta).$$

for some $B > 0$, and therefore

$$\varphi(x) = A + B\int_{x_0}^x \exp\left(-2\int_{x_0}^y \frac{b(z)}{\sigma(z)^2}\mathrm{d}z\right)\mathrm{d}y, \qquad x \in (\alpha, \beta).$$

Note that the non-uniqueness (up to shift and rescaling) of the ODE does not affect the conclusions of the theorem.

Remark 5. We would be interested to extend this result to non-continuous coefficients, however we loose the property that $\varphi \in C^2$ and we cannot use anymore Ito formula. (We might come back to this isssue when we discuss Tanaka's formula, which is an extension of Ito formula)

Remark 6. The non-degeneracy condition $\sigma(x) > 0$ is necessary, indeed we have seen that if $\sigma(x) = |x|^{\rho}$ with $\rho \in (0, 1/2)$ and $b(x) = 0$ then the SDE has no uniqueness in law.

Remark 7. Note that the function φ does not depen on α, β.

Note that the theorem says that for any one dimensional diffusion we have that if $\mathbb{P}$ is the law of a weak solution to the SDE starting in $x \in (\alpha, \beta)$, then if we let $(Z_a)_{a \geqslant 0}$ the associated BM we have that $Z_0 = \varphi(X_0) = \varphi(x) \in (\varphi(\alpha), \varphi(\beta))$ and if we assume that $\tau < \infty$ then $\sigma < \infty$ and

$$\mathbb{P}(X_{\tau} = \alpha) = \tilde{\mathbb{P}}(Z_{\sigma} = \varphi(\alpha)) = \frac{\varphi(\beta) - \varphi(x)}{\varphi(\beta) - \varphi(\alpha)}, \qquad x \in (\alpha, \beta).$$

Note also that the non-unqueness of φ does not affect this conclusion (as it should not!).

This kind of arguments can be used to study the recurrence or transience of more general processes. For example in a future exercise sheet we will see how to apply this to study recurrence of multidimensional Brownian motion and in general of Bessel processes.

Exercise 2. Using the coordinate transformation prove that the SDE has pathwise uniqueness when b is only continuous,

$$|\sigma(x) - \sigma(y)| \leqslant C|x - y|^{1/2}, \qquad x, y \in \mathbb{R}$$

and $\sigma(x) > 0$ for any $x \in \mathbb{R}$.

Next (couple of) week: equivalence of measures in a filtered probability space, Girsanov transformation, applications of Grisanov formula: Doob's transform, change of measure, weak solution to SDE via Girsanov, simulation of SDE via Girsanov formula.