SDE techniques: Girsanov's theorem

Equivalence of measures in a filtered probability space, Girsanov transformation, applications of Girsanov formula: Doob's transform, change of measure, weak solution to SDE via Girsanov.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0})$ be a filtered probability space with a right-continuous filtration and let $\mathbb{P}, \mathbb{Q}$ two probability measures on this space. Assume that $\mathbb{Q} \ll \mathbb{P}$ and define the positive martingale

$$Z_t := \mathbb{E}[H|\mathscr{F}_t], \qquad t \geqslant 0,$$

where $H = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is the Randon-Nikodym derivative of $\mathbb{Q}$ to wrt. to $\mathbb{P}$, i.e. the unique random variable $H \in L^1(\mathbb{P})$ such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(H \mathbb{1}_A)$.

Note that $(Z_t)_{t \geqslant 0}$ is uniformly integrable and $Z_\infty = \lim_{t \to \infty} Z_t = \mathbb{E}[H|\mathscr{F}_\infty]$ in $L^1(\mathbb{P})$ and a.s.

Bayes formula: if $X \in L^1(\mathbb{Q}) \cap L^1(\mathbb{P})$ and $X \in \mathscr{F}_t$ then for any $t \geqslant s$

$$\mathbb{E}_{\mathbb{Q}}[X|\mathscr{F}_s] = \frac{\mathbb{E}_{\mathbb{P}}[Z_t X|\mathscr{F}_s]}{Z_s}, \qquad \mathbb{Q}-a.s. \tag{1}$$

this is well defined when $Z_s > 0$ and note that $\mathbb{Q}(Z_s = 0) = \mathbb{E}_{\mathbb{P}}[Z_s \mathbb{1}_{Z_s = 0}] = 0$.

Define $T = \inf\{s \geqslant 0 : Z_s = 0\}$ and recall that on $T < \infty$ we have that $Z_s = 0$ for all $s \geqslant T$, then $\mathbb{Q}(T < \infty) = \mathbb{E}_{\mathbb{P}}[Z_T \mathbb{1}_{Z_T = 0}] = 0$ and if $\mathbb{P} \ll \mathbb{Q}$ we have also that $\mathbb{P}(T < \infty) = 0$ so $Z_t > 0$ for all $t > 0$ $\mathbb{P}$-a.s.

Remark 1. There is no reason in general that the martingale $(Z_t)_{t \geqslant 0}$ is continuous. Think for example to the filtration $(\mathscr{F}_t)_{t \geqslant 0}$ generated by a Poisson process $(N_t)_{t \geqslant 0}$.

We are going to **assume** all along that $(Z_t)_{t \geqslant 0}$ is continuous and that $\mathbb{P} \sim \mathbb{Q}$.

Lemma 2. *$(X_t)_{t \geqslant 0}$ is a $\mathbb{Q}$-martingale iff $(Z_t X_t)_{t \geqslant 0}$ is a $\mathbb{P}$-martingale. The same is true also for local martingales.*

Proof. We will prove only one of the directions. Assume tha ZX is a $\mathbb{P}$-martingale, then by Bayes formula (1) we have $\mathbb{E}_{\mathbb{Q}}[X_t|\mathscr{F}_s] = Z_s^{-1} \mathbb{E}_{\mathbb{P}}[Z_t X_t|\mathscr{F}_s] = X_s$ so $(X_t)_{t \geqslant 0}$ is a martingale (check that indeed X_t is in $L^1(\mathbb{Q})$ for any $t \geqslant 0$). Assume now that the stopped process $(ZX)^T$ is a $\mathbb{P}$-martingale for some stopping time T, moreover observe that for any $A \in \mathscr{F}_s$ we have for $s < t$,

$$\mathbb{E}_{\mathbb{Q}}[X_s^T \mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[Z_s^T X_s^T \mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[Z_{t \wedge T} X_t^T \mathbb{1}_A] = \mathbb{E}_{\mathbb{Q}}[X_t^T \mathbb{1}_A]$$

(the first and last equality can be obtained by considering the partition $1 = \mathbb{1}_{T < s} + \mathbb{1}_{T \geqslant s}$ and the same for t) so

$$\mathbb{E}_{\mathbb{Q}}[X_t^T|\mathscr{F}_s] = X_s^T.$$

By localization if (XZ) is a $\mathbb{P}$-local martingale this shows that X is a $\mathbb{Q}$-local martingale. In this part of the proof we just used that $\mathbb{Q} \ll \mathbb{P}$ but in order to prove the converse one has to use that $\mathbb{P} \ll \mathbb{Q}$ to have $Z_t > 0$ always. □

Assume that $(X_t)_{t\geqslant 0}$ is a continuous spositive local martingale which is almost surely $X_t > 0$ then we can define the continuous local martingale

$$L_t = \log(X_0) + \int_0^t X_s^{-1} \mathrm{d}X_s$$

and note that $(X_t)_{t\geqslant 0}$ is the solution to the SDE $\mathrm{d}X_t = X_t \mathrm{d}L_t$ and a solution Y to this equation is given by

$$Y_t = \exp\left(L_t - \frac{1}{2}[L]_t\right) = \mathscr{E}(L)_t$$

and by using Ito formula one can check that the process $(Y_t^{-1}X_t)_{t\geqslant 0}$ is constant, therefore $X_t = X_0 Y_t$. The process $\mathscr{E}(L)$ is called the stochastic exponential of the continuous local martingale L. Via the stochastic exponential we can associate a continuous local martingale L to any continuous stricly positive local martingale X.

So we can write $Z_t = \mathscr{E}(L)_t$ (which defines L given Z).

Take $(M_t)_{t\geqslant 0}$ to be a $\mathbb{P}$-local martingale and let $\tilde{M} := M - [L, M]$ then by Ito formula

$$\mathrm{d}(Z\tilde{M})_t = Z_t \mathrm{d}\tilde{M}_t - \tilde{M}_t \mathrm{d}Z_t + \mathrm{d}[Z, \tilde{M}]_t = Z_t \mathrm{d}M_t - \tilde{M}_t \mathrm{d}Z_t + \underbrace{\mathrm{d}[Z, M]_t - Z_t \mathrm{d}[L, M]_t}_{=0} = Z_t \mathrm{d}M_t - \tilde{M}_t \mathrm{d}Z_t$$

where we used that $[Z, M]_t = [Z, \tilde{M}]_t$ and that $\mathrm{d}Z_t = Z_t \mathrm{d}L_t$. Therefore $Z\tilde{M}$ is a $\mathbb{P}$-local martingale and by the previous lemma we have that $\tilde{M}$ is a $\mathbb{Q}$-local martingale. Therefore we proved that

Theorem 3. (Girsanov) *Assume $\mathbb{Q} \sim \mathbb{P}$ and define $Z = \mathscr{E}(L)$ as above. Then if M is a $\mathbb{P}$-local martingale, the process $\tilde{M} := M - [L, M]$ is a $\mathbb{Q}$-local martingale. In particular since $[M] = [\tilde{M}]$ we have that if M is a Brownian motion then $\tilde{M}$ is also a Brownian motion.*

Remark 4. Note that $\tilde{L} = L - [L]$ so we have

$$Z_t^{-1} = \exp(-L_t + [L]_t / 2) = \exp(-\tilde{L}_t - [\tilde{L}]_t / 2) = \mathscr{E}(-\tilde{L}_t)$$

and is easy to check that

$$\mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\middle|\mathscr{F}_t\right] = Z_t^{-1}, \qquad t \geqslant 0.$$

So the relation between $\mathbb{Q}$ and $\mathbb{P}$ is simmetric, indeed $\tilde{M} = M - [L, M] = M - [\tilde{L}, \tilde{M}]$ and $M = \tilde{M} - [(-\tilde{L}), \tilde{M}]$.

Remark 5. By Girsanov's theorem we see that equivalent measures agree on classifying a process as a semimartingale. Indeed if $X = X_0 + M + V$ is a $\mathbb{P}$ semimartingale then X is also a $\mathbb{Q}$-semimartingale with decomposition $X = X_0 + \tilde{M} + \tilde{V}$ where $\tilde{M} = M - [L, M]$ and $\tilde{V} = V + [L, M]$.

In many applications we have a measure $\mathbb{P}$ and a positive continuous martingale $(Z_t)_{t\geqslant 0}$ with which we can define a new measure $\mathbb{Q}$ such that

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = Z_t, \qquad t \geqslant 0.$$

This is enough to define the measure $\mathbb{Q}$ on $\mathscr{F}_\infty = \vee_{t\geqslant 0}\mathscr{F}_t$. If this is not the full $\mathscr{F}$ then we can simply let $\mathrm{d}\mathbb{Q} = Z_\infty \mathrm{d}\mathbb{P}$ where $Z_\infty = \lim_{t\to\infty} Z_t$ provided the martingale is uniformly integrable.

Note that Z is uniformly integrable iff $\mathbb{Q} \ll \mathbb{P}$.

However in many aplications we only have that $(Z_t)_{t\geqslant 0}$ is a martingale but not uniformly integrable. In that case we can apply Girsanov's theorem on any bounded interval $[0, T]$ so we can also deduce that it extends to this situation.

Example 6. (Brownian motion with drift) Let $\gamma \in \mathbb{R}^n$ and B to be a n-dimensional Brownian motion, define the process $L_t = \gamma \cdot B_t$

$$Z_t = \exp\left(L_t - \frac{1}{2}[L]_t\right) = \exp\left(\gamma \cdot B_t - \frac{1}{2}|\gamma|^2 t\right), \qquad t \geqslant 0,$$

is a strictly positive continuous local martingale and it defines a new measure $\mathbb{Q}$ on $\mathscr{F}_\infty$ under which the process

$$\tilde{B}_t^\alpha = B_t^\alpha - [L, B^\alpha]_t = B_t^\alpha - \gamma^\alpha t, \qquad \alpha = 1, \ldots, n, t \geqslant 0,$$

is a $\mathbb{Q}$-Brownian motion. So under $\mathbb{Q}$ the process B is Brownian motion with a drift γ. The measure $\mathbb{Q}$ is not absolutely continuous wrt. $\mathbb{P}$. Indeed consider the event

$$A = \left\{ \lim_{t \to \infty} \frac{\tilde{B}_t + \gamma t}{t} = \gamma \right\} \in \mathscr{F}_\infty$$

for which we have (by the law of iterated log) $\mathbb{Q}(A) = 1$ while $\mathbb{P}(A) = 0$ unless $\gamma = 0$. And similarly one shows that $\mathbb{P}, \mathbb{Q}$ are singular. This is linked to the fact that $(Z_t)_{t \geqslant 0}$ is not uniformly integrable.

Next lecture: Doob's transform, applications to weak solutions of SDE, and also to conditioning, relations with the martingale problems.
