Institute for Applied Mathematics – SS2020
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V4F1 Stochastic Analysis – Problem Sheet 1

Version 2. Tutorial classes: Mon May 4th 16–18 (Zoom) Min Liu |Wed May 7th 16–18 (Zoom) Daria Frolova. Solutions in groups of 2 (at most). To be handled in LaTeX or TeXmacs format via eCampus not later than 4pm Thursday April 30th. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

Names: XXXXXXXXXXXX/YYYYYYYYYYYYY

Exercise 1 (Pts 4+2) (Martingale problem) Let $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ locally bounded coefficients. Let $a(x) = \sigma(x)\sigma(x)^T \in \mathbb{R}^{n\times n}$ and for all $f \in C^2(\mathbb{R}^n)$ let

$$\mathcal{L}f(x) = b(x) \cdot \nabla f(x) + \frac{1}{2}\operatorname{Tr}[a(x)\nabla^2 f(x)], \qquad x \in \mathbb{R}^n$$

where $\nabla^2 f(x)$ is the $\mathbb{R}^{n\times n}$ matrix of second derivatives of f.

a) Prove that the following conditions are equivalent

  i. For any $f \in C^2(\mathbb{R}^d)$, the process $M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\mathrm{d}s$ is a local martingale.

  ii. For any $v \in \mathbb{R}^d$, the process $M_t^v = v \cdot X_t - v \cdot X_0 - \int_0^t v \cdot b(X_s)\mathrm{d}s$ is a local martingale with quadratic variation

$$[M^v]_t = \int_0^t v \cdot a(X_s)v\mathrm{d}s.$$

  iii. For any $v \in \mathbb{R}^d$ the process

$$Z_t^v = \exp\left(M_t^v - \frac{1}{2}\int_0^t v \cdot a(X_s)v\mathrm{d}s\right)$$

  is a local martingale.

[Hint: use the fact that linear combinations of exponentials are dense in C^2 w.r.t. uniform convergence on compacts for the functions and its first two derivatives (assumed wihtout proof)]

b) Show that any of conditions a,b,c imply that

$$(f(X_t)/f(X_0))\exp\left(-\int_0^t \frac{\mathcal{L}f}{f}(X_s)\mathrm{d}s\right)$$

is a local martingale for every stricly positive C^2 function f.

Exercise 2 (Pts 2+2+2) Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion. Find the SDEs satisfied by the following processes: (for all $t \geqslant 0$)

a) $X_t = B_t/(1+t)$,

b) $X_t = \sin(B_t)$

c) $(X_t, Y_t) = (a\cos(B_t), b\sin(B_t))$ where $a, b \in \mathbb{R}$ with $ab \neq 0$

Exercise 3 (Pts 2+2+2+2) (Variation of constants) Consider the nonlinear SDE

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + c(t)X_t\mathrm{d}B_t, \qquad X_0 = x,$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}_+ \to \mathbb{R}$ are continuous deterministic functions.

a) Find an explicit solution Z_t in the case $f = 0$ and $Z_0 = 1$.

b) Use the Ansatz $X_t = C_tZ_t$ to show that X solves the SDE provided C solves an ODE with random coefficients.

c) Apply this method to solve the SDE

$$\mathrm{d}X_t = X_t^{-1}\mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x$$

where α is a constant.

d) Apply the method to study the solution of the SDE

$$\mathrm{d}X_t = X_t^{\gamma}\mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x > 0$$

where α and γ are constants. For which values of γ do we get explosion ,i.e. the solution tends to $+\infty$ for finite time?
