**Institute for Applied Mathematics – SS2020**
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# V4F1 Stochastic Analysis – Problem Sheet 10

Version 1, 2020.06.29. Tutorial classes: Mon July 6th 16–18 (Zoom) Min Liu |Wed July 8th 16–18 (Zoom) Daria Frolova. Solutions in groups of 2 (at most). To be handled in LaTeX or TeXmacs format via eCampus not later than 8pm Friday July 3rd. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

**Names: XXXXXXXXXXXX/YYYYYYYYYYYYY**

Let $(\Omega := C(\mathbb{R}_{\geqslant 0};\mathbb{R}), \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ the one dimensional Wiener space and $X$ the canonical process.

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**Exercise 1 (Pts 2+2+2+2+2)** Find a predictable process $F$ such that

$$\Phi = \mathbb{E}[\Phi] + \int_0^\infty F_s \mathrm{d}X_s$$

when $\Phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is each of the following r.v. (with $T > 0$ fixed)

$$X_T^2, \quad e^{X_T}, \quad \int_0^T X_t \mathrm{d}t, \quad X_T^3, \quad \sin(X_T).$$

(One possible approach: for any $\Phi$ try to find a martingale $(M_t)_t$ such that $M_T = \Phi$, and then apply Ito formula).

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**Exercise 2 (Pts 2+2+2)** We want to prove that the linear span of r.v. of the form

$$E(h) = \cos\left(\int h_s \mathrm{d}X_s\right)\exp\left(\frac{1}{2}\int h_s^2 \mathrm{d}s\right), \quad F(h) = \sin\left(\int h_s \mathrm{d}X_s\right)\exp\left(\frac{1}{2}\int h_s^2 \mathrm{d}s\right), \qquad h \in L^2(\mathbb{R}_{\geqslant 0}),$$

is dense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ ($h$ is a deterministic function and the integrals are over $\mathbb{R}_{\geqslant 0}$).

a) Show that if $G \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is orthogonal to all $\{E(h), F(h) : h \in L^2(\mathbb{R})\}$, then in particular

$$\mathbb{E}[G \exp(i\lambda_1 B_{t_1} + \cdots + i\lambda_n B_{t_n})] = 0$$

for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $t_1, \cdots, t_n \geqslant 0$.

b) Deduce from this that $G$ is orthogonal to all functions of the from $\phi(B_{t_1}, \ldots, B_{t_n})$ with $\phi \in C_0^\infty$. [Hint: use Fourier transform]

c) Conclude.

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**Exercise 3 (Pts 4+4)** Use the class of functions introduced in Exercise 2 to reprove the Brownian martingale representation theorem.

a) Determine the martingale representation for functions $\Phi$ of the from

$$\Phi = \sum_i (a_i E(h_i) + b_i F(h_i))$$

where $a_i, b_i \in \mathbb{R}$, $h_i \in L^2(\mathbb{R}_{\geqslant 0})$ and the sum is finite.

b) Use the density of such functions to approximate an arbitrary element $\Phi \in L^2$ and conclude.

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