Institute for Applied Mathematics – SS2020
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V4F1 Stochastic Analysis – Problem Sheet 2

Version 2. Tutorial classes: Mon May 11th 16–18 (Zoom) Min Liu |Wed May 13th 16–18 (Zoom) Daria Frolova. Solutions in groups of 2 (at most). To be handled in LaTeX or TeXmacs format via eCampus not later than 4pm Thursday May 7th. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

Names: XXXXXXXXXXXX/YYYYYYYYYYYYY

Exercise 1 (Pts 2+3+2) Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion.

a) Define the process

$$X_t = a(t)\left(x_0 + \int_0^t b(s)\mathrm{d}B_s\right)$$

where $a, b : \mathbb{R}_+ \to \mathbb{R}$ are differentiable functions with $a(0) = 1$ and $a(t) > 0$. Compute the SDE satisfied by this process.

b) Use (a) to find an explicit solution for the SDEs in eqns.(1), (2), (3):

$$\begin{cases} \mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}B_t & t \in [0, T] \\ X_0 = x_0 \end{cases} \tag{1}$$

where α, σ, T are positive constants.

$$\begin{cases} \mathrm{d}X_t = -\frac{X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t & t \in [0, 1) \\ X_0 = 0 \end{cases} \tag{2}$$

$$\begin{cases} \mathrm{d}X_t = tX_t \mathrm{d}t + e^{t^2/2}\mathrm{d}B_t & t \in [0, T] \\ X_0 = 1 \end{cases} \tag{3}$$

c) Are the solutions of the SDEs in (b) strong and pathwise unique?

Exercise 2 (Pts 2+2+2) Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion.

a) Given $f \in C(\mathbb{R}_+)$, prove that $X_t = \int_0^t f(s)\mathrm{d}B_s$ is a Gaussian random variable with mean 0 and variance $\int_0^t f(u)^2 \mathrm{d}u$ for all $t \geqslant 0$.

b) The Ornstein–Uhlenbeck process $(X_t)_{t\geqslant 0}$ is defined as the solution to the SDE

$$\begin{cases} \mathrm{d}X_t = (-\alpha X_t + \beta)\mathrm{d}t + \sigma \mathrm{d}B_t & t \geqslant 0 \\ X_0 = x_0 \end{cases} \tag{4}$$

where α, σ are positive constant and $\beta, x_0 \in \mathbb{R}$. Find the explicit solution to the SDE (4).

c) Prove that X_t converges in distribution as $t \to \infty$ to a Gaussian random variable with mean β/α and variance $\sigma^2/2\alpha$.

Exercise 3 (Pts 3+2+2) Let $(B_t)_{t\geqslant 0}$ be a 2-dimensional Brownian motion and X a two-dimensional stochastic process solution to the SDE

$$\begin{cases} \mathrm{d}X_t = AX_t \mathrm{d}t + \mathrm{d}B_t & t \geqslant 0 \\ X_0 = \xi \end{cases} \tag{5}$$

where ξ is a random variable in $\mathbb{R}^2$ independent of B and

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

with $\alpha \in \mathbb{R}$.

a) Let $\phi(t)$ be a 2×2 matrix that satisfies the ODE

$$\dot{\phi}(t) = A\phi(t), \qquad \phi(0) = \mathbb{I}_2$$

where $\mathbb{I}_2$ is the 2×2 identity matrix. Show that $\phi(t) = e^{At} = \sum_{n\geqslant 0} A^n \frac{t^n}{n!}$ and calculate $\phi(t)$ explicitly. Find $\phi(t)^{-1}$ (inverse matrix).

b) Verify that

$$X_t = \phi(t)\left(\xi + \int_0^t \phi(s)^{-1} \mathrm{d}B_s\right)$$

solves the SDE (5).

c) Calculate the explicit solution of (5).
