Institute for Applied Mathematics – SS2020
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V4F1 Stochastic Analysis – Problem Sheet 3

Version 2, 2020.05.07. Tutorial classes: Mon May 18th 16–18 (Zoom) Min Liu |Wed May 20th 16–18 (Zoom) Daria Frolova.
Solutions in groups of 2 (at most). To be handled in LaTeX or TeXmacs format via eCampus not later than 4pm Thursday May 14th. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

Names: XXXXXXXXXXXX/YYYYYYYYYYYYY

Exercise 1 (Pts 3) (Constant quadratic variation) Let M be a continuous local martingale and $S \leqslant T$ two stopping times. Prove that $[M]_T = [M]_S < \infty$ a.s implies $M_t = M_S$ for all $t \in [S, T]$ a.s. . *[Hint: consider the continuous local martingale $N_t = \int_0^t \mathbb{I}_{]S,T]}(s) \mathrm{d}M_s$].*

Exercise 2 (Pts 3+3) (Feynman–Kac formula for Ito diffusions)

a) Consider the solution X of the SDE in $\mathbb{R}^n$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \qquad X_0 = x,$$

where B is a d-dimensional Brownian motion and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ locally bounded continuous coefficients. Let $\mathcal{L}$ be the associated infinitesimal generator. Fix $t > 0$ and assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $V : [0, t] \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ are continuous functions. Show that any bounded $C^{1,2}$ solution $u : [0, t] \times \mathbb{R}^n \to \mathbb{R}$ of the equation

$$\begin{array}{rcll} \frac{\partial}{\partial s} u(s,x) &=& \mathcal{L}u(s,x) - V(s,x)u(s,x), & \qquad (s,x) \in (0,t] \times \mathbb{R}^n, \\ u(0,x) &=& \varphi(x), & \end{array}$$

has the stochastic representation

$$u(t,x) = \mathbb{E}\left[\varphi(X_t) \exp\left(-\int_0^t V(t-s, X_s)\mathrm{d}s\right)\right].$$

In particular, there is at most only one solution of the PDE.
[Hint: show that $M_r = \exp\left(-\int_0^r V(t-s, X_s)\mathrm{d}s\right) u(t-r, X_r)$ is a local martingale].

b) The price of a security is modeled by a geometric Brownian motion X with parameters $\alpha, \sigma > 0$:

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t, \qquad X_0 = x > 0.$$

At price y we have a running cost of $V(y)$ per unit time. The total cost up to time t is then

$$A_t = \int_0^t V(X_s)\mathrm{d}s.$$

Suppose that u is a bounded solution to the PDE

$$\begin{array}{rcll} \frac{\partial}{\partial s} u(s,x) &=& \mathcal{L}u(s,x) - \beta V(x)u(s,x), & \qquad (s,x) \in (0,t] \times \mathbb{R}_{\geqslant 0}, \\ u(0,x) &=& 1, & \end{array}$$

where $\mathcal{L}$ is the generator of X. Show that the Laplace transform of A_t is given by

$$\mathbb{E}[e^{-\beta A_t}] = u(t,x).$$

Exercise 3 (Pts 3+3+3+2) (Continuous Branching Process) Consider a family of diffusions $(X_t(x))_{t>0,x>0}$ satisfying the SDE

$$\mathrm{d}X_t(x) = \alpha X_t(x)\mathrm{d}t + \sqrt{\beta X_t(x)}\mathrm{d}B_t, \qquad X_0(x) = x,$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}_{>0}$. Existence of strong solutions to this equation follows from the Yamada–Watanabe theorem. Let $(\tilde{X}, \tilde{B})$ be an independent copy of (X, B) and let $Y_t(x, y) = X_t(x) + \tilde{X}_t(y)$ for $t > 0, x > 0, y > 0$.

a) *(Branching)* Compute the SDE satisfied by Y and prove that $(Y(x,y))_{t\geqslant 0}$ has the same law of $(X_t(x+y))_{t\geqslant 0}$. *[Hint: use martingale caracterization of weak solutions and pathwise uniqueness]*

b) *(Duality)* Show that this implies that there exists a function $u : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ such that

$$\mathbb{E}[e^{-\lambda X_t(x)}] = e^{-xu(t,\lambda)}, \qquad x \in \mathbb{R}_{>0} \tag{1}$$

if we assume that the map $x \mapsto \mathbb{E}[e^{-\lambda X_t(x)}]$ is continuous.

c) Assume that $u : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ is differentiable with respect to its first parameter. Apply Ito formula to $s \mapsto G_s = e^{-u(t-s,\lambda)X_s(x)}$ and determine which differential equation u should satisfy in order for G to be a local martingale. Prove that in this case eq. (1) is satisfied (in particular, if a solution of the equation exists then it is unique).

d) *(Extinction probability)* Find the explicit solution u for the differential equation and using eq. (1) prove that if $\alpha = 0$ then

$$\mathbb{P}(X_t(x) = 0) = e^{-2x/(\beta t)}, \qquad x, t > 0.$$
