Institute for Applied Mathematics – SS2020
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V4F1 Stochastic Analysis – Problem Sheet 4

Version 1, 2020.05.12. Tutorial classes: Mon May 25th 16–18 (Zoom) Min Liu |Wed May 27th 16–18 (Zoom) Daria Frolova.
Solutions in groups of 2 (at most). To be handled in LaTeX or TeXmacs format via eCampus not later than 4pm Thursday May 21th. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

Names: XXXXXXXXXXXX/YYYYYYYYYYYYY

Exercise 1 (Pts 2) (Brownian motion on the unit sphere) Let $Y_t = B_t / |B_t|$ where B is a Brownian motion in $\mathbb{R}^n$ and $n > 2$. Prove that the time–changed process

$$Z_a = Y_{T_a}, \qquad T = A^{-1}, \qquad A_t = \int_0^t |B_s|^{-2} \mathrm{d}s,$$

is a diffusion taking values in the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with generator

$$\mathcal{L}f(x) = \frac{1}{2}\left(\Delta f(x) - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right) - \frac{n-1}{2}\sum_i x_i \frac{\partial f}{\partial x_i}(x), \qquad x \in S^{n-1}.$$

where Δ is the Laplacian in $\mathbb{R}^n$ and where diffusion here means continuous time process solving the martingale problem for this generator.

Exercise 2 (Pts 2+2+2+1+1) (Polar points of Brownian motion for $d \geqslant 2$) Let (X, Y) be a Brownian motion on $\mathbb{R}^2$ starting at $(0, 0)$. Let

$$(M_t, N_t) := e^{X_t}(\cos(Y_t), \sin(Y_t)).$$

We will assume without proof that

$$\int_0^\infty e^{2X_s} \mathrm{d}s = +\infty, \qquad a.s.$$

a) Prove that (M, N) is a Brownian motion on $\mathbb{R}^2$ changed of time (starting from where?) ;

b) Compute the Euclidean norm $|(M_t, N_t)|$ of the vector (M_t, N_t) and deduce that a Brownian motion B in $\mathbb{R}^2$ never visit the point $(-1, 0)$, that is

$$\mathbb{P}(\exists t > 0 : B(t) = (-1, 0)) = 0.$$

c) Conclude that B never visit any given point $x \neq (0, 0)$.

d) Use the Markov property to deduce from (c) that $\mathbb{P}(\exists t > 0 : B(t) = (0, 0)) = 0$. [Hint: consider $\mathbb{P}(\exists t \geqslant 1/n : B(t) = (0, 0))$ as $n \to 0$.]

e) Prove that a Brownian motion in $\mathbb{R}^d$ with $d > 2$ does not visit any given point $x \in \mathbb{R}^d$.

Exercise 3 (Pts 2+2+2+1+1) (Transience of Brownian motion in $d \geqslant 3$) Let X be a Brownian motion in $\mathbb{R}^3$ starting from $a \in \mathbb{R}^3 \neq 0$. We say that a process Y is transient if $|Y_t| \to \infty$ as $t \to \infty$ almost surely.

a) Prove that the process $M_t = 1/|X_t|$ is a positive local martingale.

b) Prove that $M_\infty = \lim_{t\to\infty} M_t$ exists almost surely.

c) Compute $\mathbb{E}[M_t]$ and deduce that $M_\infty = 0$. This implies that X is transient.

d) Show that whatever the starting point is, X is always transient.

e) Prove that a Brownian motion in $\mathbb{R}^d$ with $d \geqslant 3$ is transient.

Exercise 4 (Pts 2) (Conformal invariance of Brownian motion) Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function and $Z = X + iY$ be a planar Brownian motion (with the identification of $\mathbb{C}$ with $\mathbb{R}^2$). Prove that the process $M_t = f(Z_t)$ is a continuous local martingale with values in $\mathbb{C}$. Deduce that it is a complex Brownian motion changed of time. This property is called conformal invariance of Brownian motion.