Institute for Applied Mathematics – SS2020
Massimiliano Gubinelli

V4F1 Stochastic Analysis – Problem Sheet 5

Version 1, 2020.05.22. Tutorial classes: Mon June 1st 16–18 (Zoom) Min Liu |Wed June 3rd 16–18 (Zoom) Daria Frolova.
Solutions in groups of 2 (at most). To be handled in LaTeX or TeXmacs format via eCampus not later than 4pm Thursday May 28th. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

Names: XXXXXXXXXXXX/YYYYYYYYYYYYY

Exercise 1 (Pts 2+2+2) (Passage time to a sloping line) Let X be a one–dimensional Brownian motion with $X_0 = 0$ and let $a > 0$, $b \in \mathbb{R}$.

a) Let $T_L = \inf\{t \geqslant 0 : X_t = a + bt\}$ denote the first passage time to the line $y = a + bt$. Show that

$$\mathbb{P}(T_L \leqslant t) = \mathbb{E}[e^{-bX_t - b^2 t/2} \mathbb{I}_{T_a \leqslant t}], \tag{1}$$

where $T_a = \inf\{t \geqslant 0 : X_t = a\}$ is the first passage time to level a.

b) Recall that, by the reflection principle, the law of T_a is absolutely continuous with density

$$f_{T_a}(t) = a t^{-3/2} \varphi\left(a/\sqrt{t}\right) \mathbb{I}_{(0,\infty)}(t),$$

where φ is the standard normal density. Deduce that the law of T_L is absolutely continuous with density

$$f_{T_L}(t) = a t^{-3/2} \varphi\left((a + bt)/\sqrt{t}\right) \mathbb{I}_{(0,\infty)}(t).$$

[Hint: in (1) *take the conditional expectation w.r.t.* $\mathcal{F}_{T_a}$*].*

c) Show that, for $b > 0$,

$$\mathbb{E}[e^{-bX_t} \max_{s \leqslant t}(X_s)] \simeq \frac{e^{b^2 t/2}}{2b}, \quad \text{and} \quad \mathbb{E}[e^{bX_t} \max_{s \leqslant t}(X_s)] \simeq b\, t e^{b^2 t/2}, \qquad \text{as } t \to \infty.$$

Exercise 2 (Pts 2+2+3) (Brownian Bridge) Let X be a d–dimensional Brownian motion with $X_0 = 0$.

a) Show that, for any $y \in \mathbb{R}^d$, the process

$$X_t^y = X_t - t(X_1 - y) \qquad t \in [0, 1]$$

is independent of X_1.

b) Let μ_y denote the law of X^y on $C([0, 1]; \mathbb{R}^d)$. Show that $y \mapsto \mu_y$ is a regular version of the conditional distribution of X given $X_1 = y$.

c) Compute the SDE satisfied by the canonical process Y under the probability measure μ_y on the space $C([0, 1]; \mathbb{R}^d)$. (Hint: use Doob's h-transform argument from the lectures)

Exercise 3 (Pts 3) Let M be a positive continuous supermartingale such that $\mathbb{E}[M_0] < \infty$. Let $M_\infty = \lim_{t \to \infty} M_t$. Show that if $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$ then M is a martingale and $\mathbb{E}[M_\infty | \mathcal{F}_t] = M_t$. *[Hint: prove that* $\mathbb{E}[M_\infty | \mathcal{F}_t] \leqslant M_t$ *and that* $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ *and conclude.]*

Exercise 4 (Pts 4) Prove directly that the h-transform gives a transformation of martingale problems from the one with drift b and diffusion σ to another with same diffusion coefficient σ but different drift $\tilde{b}$.
