

## V4F1 Stochastic Analysis – Problem Sheet 8

Version 1, 2020.06.10. Tutorial classes: Mon June 22nd 16–18 (Zoom) Min Liu | Wed June 24th 16–18 (Zoom) Daria Frolova.  
Solutions in groups of 2 (at most). To be handled in L<sup>A</sup>T<sub>E</sub>X or T<sub>E</sub>X<sub>M</sub>A<sub>C</sub>S format via eCampus not later than 8pm Thursday June 18th. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

Names: XXXXXXXXXXXX/YYYYYYYYYYYYYYYY

---

**Exercise 1 (Pts 2+2+2+2)** Assume that  $\Omega = C(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$ ,  $\mathbb{P}$  is the  $d$ -dimensional Wiener measure and that  $X$  is the canonical process on  $\Omega$  and that the filtration  $\mathcal{F}_\bullet$  is generated by  $X$ . Consider a predictable  $\mathbb{R}^d$ -valued drift  $b$  given by a function  $b : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}^d$ . By tilting  $\mathbb{P}$  via  $Z = \mathcal{E}(\int_0^\cdot b(X)dX)$  we obtain that, under the tilted measure  $\mathbb{P}^b$  the process  $X$  is a solution of the SDE

$$dX_t = b_t(X) + dW_t, \quad t \geq 0$$

where  $W$  is a  $\mathbb{P}^b$ -Brownian motion.

a) Prove that if

$$|b_t(x)| \leq C(1 + |x_t|), \quad t \geq 0, x \in \Omega,$$

then Novikov's condition holds conditionally on  $\mathcal{F}_s$  for intervals  $[s, t]$  such that  $|t - s|$  is small enough, i.e.

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_s^t |b_u(X)|^2 du \right) \middle| \mathcal{F}_s \right] < +\infty.$$

b) Deduce that  $Z$  is a martingale. [Hint: prove that  $\mathbb{E}[Z_t | \mathcal{F}_s] = Z_s$  for small time intervals  $[s, t]$  and the conclude].

c) Prove that

$$\mathbb{P}(\|X\|_{[0,t]} > r) \leq 2de^{-r^2/2dt} \quad t \geq 0, r \geq 0.$$

where  $\|X\|_{[0,t]}$  denotes the supremum wrt. the Euclidean norm of  $(X_s)_{s \in [0,t]}$ .

[Hint: use Doob's inequality for the submartingale  $e^{\lambda X_t^i}$  and optimize over  $\lambda > 0$ ]

d) Prove the same result as in (a) under the more general assumption that  $b$  is a previsible drift such that

$$|b_t(x)| \leq C(1 + \|x\|_{\infty, [0,t]}), \quad t \geq 0, x \in \Omega$$

where  $C < +\infty$ .

---

**Exercise 2 (Pts 2+2+2)** Consider the one dimensional SDE

$$dX_t = -X_t^3 dt + dB_t, \quad X_0 = x,$$

where  $B$  is a standard Brownian motion.

a) Let  $f(t, x) = (1 + |x|^2)$  and  $T_L = \inf\{t \geq 0 : |X_t| > L\}$ . Use Ito formula to show that there exists a constant  $\lambda$  such that the process  $Z_t := e^{-\lambda(t \wedge T_L)} f(X_{t \wedge T_L})$  is a supermartingale.

b) Deduce that  $\mathbb{P}(T_L \leq t) \rightarrow 0$  as  $L \rightarrow \infty$ .

c) Conclude that solutions of the SDE cannot explode (that is  $\zeta := \sup_L T_L = \infty$  a.s.).

---

**Exercise 3 (Pts 2+2+2)** If  $c(t) = (x(t), y(t))$  is a smooth curve in  $\mathbb{R}^2$  with  $c(0) = 0$ ,

$$A_t = \int_0^t (x(s)y'(s) - y(s)x'(s))ds$$

describes the area that is covered by the secant from the origin to  $c(s)$  in the interval  $[0, t]$ . Analogously, for a two-dimensional Brownian motion  $B_t = (X_t, Y_t)$  with  $B_0 = 0$ , one defines the Lévy Area

$$A_t = \int_0^t (X_s dY_s - Y_s dX_s).$$

a) Let  $\alpha(t), \beta(t)$  be  $C^1$ -functions,  $p \in \mathbb{R}$ , and

$$V_t = ipA_t - \frac{\alpha(t)}{2}(X_t^2 + Y_t^2) + \beta(t).$$

Use Itô formula to show that  $e^{V_t}$  is a local martingale provided  $\alpha'(t) = \alpha(t)^2 - p^2$  and  $\beta'(t) = \alpha(t)$

b) Let  $t_0 \geq 0$ . Solutions to the equations for  $\alpha, \beta$  with  $\alpha(t_0) = \beta(t_0) = 0$  are

$$\alpha(t) = p \tanh(p(t_0 - t)), \quad \beta(t) = -\log \cosh(p(t_0 - t)).$$

Conclude that

$$\mathbb{E} \left[ e^{ipA_{t_0}} \right] = (\cosh(pt_0))^{-1}.$$

c) Show that the distribution of  $A_t$  is absolutely continuous with respect to the Lebesgue measure with density

$$f_{A_t}(x) = (2t \cosh(\pi x/2t))^{-1}, \quad x \in \mathbb{R}.$$

---