Institute for Applied Mathematics – SS2020
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V4F1 Stochastic Analysis – Problem Sheet 9

Version 1, 2020.06.19. Tutorial classes: Mon June 29th 16–18 (Zoom) Min Liu |Wed July 1st 16–18 (Zoom) Daria Frolova.
Solutions in groups of 2 (at most). To be handled in LaTeX or TeXmacs format via eCampus not later than 8pm Friday June 26th. Use this sheet for your solutions and write them under the corresponding exercise. Fill out your names below.

Names: XXXXXXXXXXXX/YYYYYYYYYYYYY

Exercise 1 (Pts 4) Prove that if B is a Brownian motion, then we have the relation $L_t^{|B|,0} = 2L_t^{B,0}$ where $L_t^{X,a}$ denotes the local time in $a \in \mathbb{R}$ of the semimartingale X.

Exercise 2 (Pts 2+4) Let $y : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function and let

$$a(t) = \sup_{s \in [0,t]} (y(s))_- = \sup_{s \in [0,t]} (-y(s) \vee 0), \qquad z(t) = y(t) + a(t).$$

a) Prove that a, z are continuous functions and a is non-decreasing.
b) Prove that a is of bounded variation and that $\int_0^\infty \mathbb{1}_{z_s > 0} \mathrm{d}a_s = 0$. (Hint: use the fact that $\mathrm{d}a_s$ is a Borel measure).

Exercise 3 [Pts 2+2+2] Prove (the upper bound of) Burkholder–Davis–Gundy inequality. Let M be a continuous local martingale (with $M_0 = 0$). For any $p \geqslant 2$ we have

$$\mathbb{E}[\sup_{t \leqslant T} |M_t|^p] \leqslant C_p \mathbb{E}[([M]_T^{p/2})]$$

where C_p is a universal constant depending only on p.

a) Assume that the martingale M is bounded. Use Itô formula on $t \mapsto (\varepsilon + |M_t|^2)^{p/2}$ to prove that

$$\mathbb{E}[\sup_{t \leqslant T} |M_t|^p] \leqslant \int_0^T \mathbb{E}[|M_t|^{p-2} \mathrm{d}[M]_t].$$

(why we need $\varepsilon > 0$?)

b) Use Hölder's and Doob's inequality to conclude.
c) Remove the assumption of boundedness.

Exercise 4 (Pts 2+2+2) Let us continue with the setting of Exercise 3 and prove now a complementary lower bound when $p \geqslant 4$, that is

$$\mathbb{E}[([M]_T^{p/2})] \leqslant C_p \mathbb{E}[\sup_{t \leqslant T} |M_t|^p].$$

where again C_p is a universal constant depending only on p (not the same as that of the upper bound).

a) Use the relation

$$[M]_T = M_T^2 - 2 \int_0^T M_s \mathrm{d}M_s$$

to estimate $\mathbb{E}[([M]_T^{p/2})]$ and then use the BDG upper bound for the stochastic integral.

b) Prove that if we let $N_T = \int_0^T M_s \mathrm{d}M_s$ then for any $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that

$$[N]_T^{1/2} \leqslant \lambda_\varepsilon \sup_{t \leqslant T} |M_t| + \varepsilon [M]_T$$

c) Conclude by choosing ε small enough.
