Foundations in Stochastic Analysis

WS 2021/22

31.01.2022

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Chapter 1

Brownian motion, Poisson process and Levy processes

1.1 Definition and equivalent characterizations of Brownian motion

Definition 1.1. *A stochastic process* $B: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ *is a Brownian motion if*

1. $B_0 = 0$,
2. *for any* $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \in \mathbb{R}_+$ *we have that* $B_{t_1} - B_0$, $B_{t_2} - B_{t_1}, \ldots,$ $B_{t_n} - B_{t_{n-1}}$ *are independent random variables and* $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$,
3. *for almost every* $\omega \in \Omega$ *the function* $t \longmapsto B_t(\omega)$ *is continuous (i.e., in* $C^0(\mathbb{R}_+, \mathbb{R})$ *).*

1.1.1 Brownian motion as a Markov process

We consider the following completed natural filtration of B_t given by

$$\mathcal{F}_t = \sigma(B_s, s \in [0, t]).$$

Theorem 1.2. *A Brownian motion* B_t *is a* $\mathcal{F}_t$ *Markov process with transition kernel given by*

$$p(x, t; y, s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right), \tag{1.1}$$

where $0 \leq s < t$.

Proof. We have to prove that for any $0 \leq s < t$ and any Borel set $A \subset \mathbb{R}$ there exists a version of $\mathbb{P}(B_t \in A | \mathcal{F}_s)$ which is $\sigma(B_s)$ measurable.

By Definition 1.1 we have that $B_t - B_s$ is independent of $B_s - B_0 = B_s$ and $B_t - B_s \sim N(0, t-s)$

$$\begin{aligned} \mathbb{P}(B_t \in A | \mathcal{F}_s) &= \mathbb{P}((B_t - B_s) + B_s \in A | \mathcal{F}_s) \\ &= \int_A \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(\frac{-(x-B_s)^2}{2(t-s)}\right). \end{aligned}$$

□

Corollary 1.3. *For any* $0 < t_1 < t_2 < \cdots < t_n$ *we have that the law of* $(B_{t_1}, \ldots, B_{t_n})$ *is given by*

$$\frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \exp\left(-\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right), \tag{1.2}$$

where $t_0 = 0$ *and* $x_0 = 0$.

Proof. We prove the theorem for $n = 2$. The general case can be proved by induction.

Let A_1, A_2 be two Borel subsets of $\mathbb{R}$, then we have

$$\begin{aligned} \mathbb{P}(B_{t_1} \in A_1, B_{t_2} \in A_2) &= \int_{A_1} \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x_1^2}{2t_1}\right) \mathbb{P}(B_{t_2} \in A_2 | B_{t_1} = x_1) \mathrm{d}x_1 \\ &= \int_{A_1} \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x_1^2}{2t_1}\right) \left(\int_{A_2} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left(\frac{-(x_2 - x_1)^2}{2(t_2 - t_1)}\right) \mathrm{d}x_2\right) \mathrm{d}x_1 \end{aligned}$$

where to obtain the last equality we use Theorem 1.2. □

Corollary 1.4. *Let B_t be a Markov process with transition kernel (1.1), $B_0=0$ and such that for almost every $\omega\in\Omega$ the function $t\longmapsto B_t(\omega)$ is in $C^0(\mathbb{R}_+,\mathbb{R})$, then B_t is a Brownian motion.*

Proof. We have only to prove that B_t satisfies the second property of Definition 1.1. Using the same reasoning of Corollary 1.3, we obtain that, if B_t is a Markov process with transition kernel (1.1), then it has finite dimensional marginals given by (1.2). This implies that for any $0\leq t_1\leq t_2\leq\ldots\leq t_n\in\mathbb{R}_+$ we have that $B_{t_1}-B_0$, $B_{t_2}-B_{t_1},\ldots$, $B_{t_n}-B_{t_{n-1}}$ are independent random variables and $B_{t_i}-B_{t_{i-1}}\sim N(0,t_i-t_{i-1})$. □

1.1.2 Brownian motion as a Gaussian process

Theorem 1.5. *Brownian motion is a Gaussian process such that $B_0=0$ and*

$$\begin{aligned}\mathbb{E}[B_t] &= 0 &(1.3)\\ \operatorname{cov}(B_t,B_s) &= \min(t,s). &(1.4)\end{aligned}$$

Proof. The fact that Brownian motion is a Gaussian process follows by the explicit expression of finite dimensional marginals given in Corollary 1.3.

Using the definition of Brownian motion we have $\mathbb{E}[B_t]=\mathbb{E}[B_t-B_0]=0$ and, if $s\leq t$,

$$\operatorname{cov}(B_t,B_s)=\operatorname{cov}(B_t-B_s,B_s)+\operatorname{cov}(B_s,B_s)=s.$$ □

Corollary 1.6. *Let B_t be a Gaussian process with mean (1.3) and co-variance (1.4), and suppose that $B_0=0$ and for almost every $\omega\in\Omega$ the function $t\longmapsto B_t(\omega)$ is in $C^0(\mathbb{R}_+,\mathbb{R})$, then B_t is a Brownian motion.*

Proof. We have only to prove that B_t satisfies the second property of Definition 1.1. Since $B_{t_1}-B_0$, $B_{t_2}-B_{t_1},\ldots$, $B_{t_n}-B_{t_{n-1}}$ are Gaussian random variables (being linear combinations of jointly Gaussian random variables) we have to prove that $\operatorname{cov}(B_{t_i}-B_{t_{i-1}},B_{t_j}-B_{t_{j-1}})=0$ if $i\neq j$. Suppose that $t_j<t_i$ then

$$\begin{aligned}\operatorname{cov}(B_{t_i}-B_{t_{i-1}},B_{t_j}-B_{t_{j-1}}) &= \operatorname{cov}(B_{t_i},B_{t_j})-\operatorname{cov}(B_{t_{i-1}},B_{t_j})-\operatorname{cov}(B_{t_i},B_{t_{j-1}})+\operatorname{cov}(B_{t_{i-1}},B_{t_j})\\ &= t_j-t_j-t_{j-1}+t_{j-1}=0,\end{aligned}$$

which concludes the proof. □

1.2 Lévy construction of Brownian motion

1.2.1 Haar and Schauder functions

We define Haar functions $h_n^k(t)$ for $n=0,1,\ldots\in\mathbb{N}$ and $k=0,\ldots,2^{n-1}-1$ in the following way: for $n=0$ we put $h_0^0(t)=1$ and for $n\neq 0$ we write

$$h_n^k(t)=2^{\frac{n-1}{2}}\Big(\mathbb{I}_{\left[\frac{2k}{2^n},\frac{2k+1}{2^n}\right)}(t)-\mathbb{I}_{\left[\frac{2k+1}{2^n},\frac{2k+2}{2^n}\right)}(t)\Big).$$

We define also Schauder functions as

$$e_n^k(t)=\int_0^s h_n^s(s)\mathrm{d}s.$$

Lemma 1.7. *The set of Haar functions forms an orthonormal basis of $L^2([0,1])$.*

Proof. The orthonormality is a consequence of the fact that $h_n^k(t)$ and $h_n^{k'}(t)$ are supported in different sets when $k\neq k'$, and that $h_n^k(t)$ has integral 0 on the dyadic set of the form $\left[\frac{k'}{2^{n-1}},\frac{k'+1}{2^{n-1}}\right]$ (for any $k'\in\mathbb{N}$).

In order to prove that the Haar functions form a complete basis of $L^2([0,1])$ we have only to prove that for any function $f\in L^2([0,1])$ such that $\int_0^1 f(t)h_n^k(t)=0$ we have $f=0$.

Consider the probability space $([0,1],\mathcal{B},\mathrm{d}x)$ (where $\mathcal{B}$ is the complete σ-algebra generated by Borel sets and $\mathrm{d}x$ is the Lebesgue measure) and consider the filtration $\mathcal{B}_n=\left\{\left[\frac{k}{2^n},\frac{k+1}{2^n}\right], k=0,\ldots,2^n-1\right\}$, with $n\in\mathbb{N}$. It is clear that $\sigma(\mathcal{B}_n|n\in\mathbb{N})=\mathcal{B}$. If $\int_0^1 f(t)h_n^k(t)=0$ for $n\leq N$ then $\int_{\left[\frac{k}{2^n},\frac{k+1}{2^n}\right]} f(t)=0$ for $n\leq N$. This implies that

$$f_n=\mathbb{E}[f|\mathcal{B}_n]=0.$$

On the other hand $\int_0^1 f_n^2(t)\mathrm{d}t=0$ and so f_n is a $\mathcal{B}_n$ martingale bounded in $L^2([0,1])$. Thus, by Doob Convergence Theorem for martingales, we have that $f_n\to\mathbb{E}[f|\mathcal{B}]=f$ in $L^1([0,1])$. This implies that $f=\lim f_n=0$. □

Lemma 1.8. *We have that* $\sup_{t\in[0,1]}|e_n^k(t)|\leq 2^{-\frac{n-1}{2}}$ *and the series*

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^{n-1}-1} e_n^k(t)e_n^k(s)\right)=\min(t,s) \tag{1.5}$$

is absolutely convergent and it is equal to $\min(t,s)$.

Proof. The bound on $|e_n^k(t)|$ follows by a direct computation. In order to prove equality (1.5) we note that $\int_0^1\mathbb{I}_{[0,t]}(\tau)h_n^k(\tau)\mathrm{d}\tau=e_n^k(t)$ (and a similar relation holds for $e_n^k(s)$). Using Parseval identity for orthonormal bases in an Hilbert space we obtain

$$\begin{aligned}\min(t,s) &= \int_0^1\mathbb{I}_{[0,t]}(\tau)\mathbb{I}_{[0,s]}(\tau)\mathrm{d}\tau\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^{n-1}-1}\int_0^1\mathbb{I}_{[0,t]}(\tau)h_n^k(\tau)\mathrm{d}\tau\int_0^1\mathbb{I}_{[0,s]}(\tau)h_n^k(\tau)\mathrm{d}\tau\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^{n-1}-1}e_n^k(t)e_n^k(s)\right)\end{aligned}$$

and the previous series is absolutely convergent. □

1.2.2 Lévy construction of Brownian motion

Let $Z_{n,k}(\omega)$ be a sequence of independent random variables such that $Z_{n,k}\sim N(0,1)$. Consider the following sequence of stochastic processes

$$B_t^N(\omega)=\sum_{n=0}^{N}\left(\sum_{k=0}^{2^{n-1}-1}Z_{n,k}(\omega)e_n^k(t)\right).$$

From now on we restrict Definition 1.1, to processes of the form $B\colon[0,1]\times\Omega\to\mathbb{R}$, i.e., defined only on the set $[0,1]$ and not on the whole positive real line $\mathbb{R}_+$.

If we have a sequence of independent Brownian motions $\tilde{B}_t^1,\ldots,\tilde{B}_t^n$ defined on $[0,1]$, we can easily build a Brownian motion B_t defined on the whole real positive line $\mathbb{R}_+$ in the following way: if $n-1<t\leq n$ (where $n\in\mathbb{N}$) we define $B_t=\sum_{k=1}^{n-1}B_1^k+B_{t-n+1}^n$.

Theorem 1.9. *The sequence of stochastic processes* B_t^N *is almost surely convergent on* $[0,1]$. *Let* B_t *be the limit of* B_t^N, *then* B_t *is a Brownian motion on* $[0,1]$.

Proof. First we prove that the sequence of functions $t\longmapsto B_t^N(\omega)$ is uniformly convergent in $C^0([0,1],\mathbb{R})$ for almost every $\omega\in\Omega$. In order to prove this, we use Weierstrass criterion for uniform convergence in $C^0([0,1],\mathbb{R})$, proving that, writing $K_n(\omega)=\sup_{t\in[0,1]}\left|\sum_{k=0}^{2^{n-1}-1}Z_{n,k}(\omega)e_n^k(t)\right|$, we have $\sum_{n=0}^{\infty}K_n<+\infty$ almost surely.

Using the fact that for fixed n the functions $e_n^k(t)$ have disjoint support, and exploiting the bound $\sup_{t\in[0,1]}|e_n^k(t)|\leq 2^{-\frac{n-1}{2}}$, we have that

$$K_n(\omega)\leq 2^{-\frac{n-1}{2}}\sup_k|Z_{n,k}(\omega)|.$$

We want to prove that there exists a positive random variable $C:\Omega\to\mathbb{R}$, almost surely finite, such that

$$\sup_k|Z_{n,k}(\omega)|\leq nC(\omega).$$

Define $\mathfrak{B}_n=\{\omega|\sup_k|Z_{n,k}(\omega)|>n\}$ then $C(\omega)<+\infty$ whenever $\omega\notin\operatorname{limsup}_n\mathfrak{B}_n$. If we are able to prove that $\mathbb{P}(\operatorname{limsup}_n\mathfrak{B}_n)=0$ then $C(\omega)<+\infty$ almost surely. In order to prove that $\mathbb{P}(\operatorname{limsup}_n\mathfrak{B}_n)=0$, we use Borel-Cantelli Lemma and the fact that $\sum_n\mathbb{P}(\mathfrak{B}_n)<+\infty$.

Indeed

$$\mathbb{P}(\mathfrak{B}_n)\leq\sum_{k=0}^{2^{n-1}-1}\mathbb{P}(|Z_{n,k}(\omega)|>n)\leq\frac{2^n}{n\sqrt{2\pi}}\exp\left(\frac{-n^2}{2}\right)$$

where we used the fact that $Z_{n,k}\sim N(0,1)$. This implies that

$$\sum_n\mathbb{P}(\mathfrak{B}_n)\leq\sum_n\frac{2^n}{n\sqrt{2\pi}}\exp\left(\frac{-n^2}{2}\right)<+\infty$$

which means that $C<+\infty$ almost surely. On the other hand we have that $K_n(\omega)\leq 2^{-\frac{n-1}{2}}\sup_k|Z_{n,k}(\omega)|$ and so

$$\sum_n K_n(\omega)\leq\sum_n 2^{-\frac{n-1}{2}}\sup_k|Z_{n,k}(\omega)|\leq C(\omega)\sum_n n2^{-\frac{n-1}{2}}<+\infty.$$

Thus the sequence $B_t^N(\omega)$ is almost surely convergent in $C^0([0,1],\mathbb{R})$.

Let B_t denote the limit of B_t^N when B_t^N is convergent and 0 otherwise. We have that B_t satisfies the condition 1 and 3 of Definition 1.1. In order to prove that B_t satisfies property 2 of Definition 1.1 we prove that B_t is a Gaussian process such that $\mathbb{E}[B_t]=0$ and $\operatorname{cov}(B_t,B_s)=\min(s,t)$. Using Corollary 1.6, this is equivalent to prove that B_t is a Brownian motion.

First we prove that for any $t\in[0,1]$ the sequence of random variables B_t^N converges to B_t in $L^2(\Omega)$. Since B_t^N converges to B_t almost surely it is sufficient to prove that B_t^N forms a Cauchy sequence in $L^2(\Omega)$. We have that

$$\begin{aligned}\mathbb{E}[(B_t^N-B_t^M)^2] &= \mathbb{E}\left[\left(\sum_{n=M}^{N}\left(\sum_{k=0}^{2^{n-1}-1}Z_{n,k}(\omega)e_n^k(t)\right)\right)^2\right]\\ &= \sum_{n=M}^{N}(e_n^k(t))^2\end{aligned}$$

when $M\leq N$ and using the fact that $Z_{n,k}$ are i.i.d. normal random variables with variance 1. On the other hand, by Lemma 1.8, the series $\sum_{n=0}^{+\infty}(e_n^k(t))^2=t<+\infty$ is absolutely convergent, this means that

$$\lim_{M\to\infty}\sum_{n=M}^{N}(e_n^k(t))^2=0,$$

which implies that B_t^N is a Cauchy sequence in $L^2(\Omega)$.

The fact that $(B_{t_1}^N,\ldots,B_{t_n}^N)$ converges to $(B_{t_1},\ldots,B_{t_n})$ in $L^2(\Omega)$ implies that B_t is a normal stochastic process (being the L^2 limit of a normal stochastic process), with $\mathbb{E}[B_t]=\lim_N\mathbb{E}[B_t^N]$ and $\operatorname{cov}(B_t,B_s)=\lim_N\operatorname{cov}(B_t^N,B_s^N)$. On the other hand we have that $\lim_N\mathbb{E}[B_t^N]=\lim_N 0=0$ and, by Lemma 1.8,

$$\lim_N\operatorname{cov}(B_t^N,B_s^N)=\lim_N\mathbb{E}[B_t^NB_s^N]=\lim_N\sum_{n=0}^{N}\left(\sum_{k=0}^{2^{n-1}-1}e_n^k(t)e_n^k(s)\right)=\min(t,s).\qquad\square$$

1.3 Definition Poisson process

Definition 1.10. *Let $\{N_t\}_{t \in \mathbb{R}_+}$ be a stochastic process, we say that N_t is a Poisson process of parameter $\lambda > 0$ if*

1. $N_0 = 0$,
2. *for any $t_1 \leqslant \cdots \leqslant t_n$ we have that $N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent and distributed as*
$$N_{t_i} - N_{t_{i-1}} \sim \mathrm{Po}(\lambda(t_i - t_{i-1}));$$
3. *the paths of N_t (namely for any $\omega \in \Omega$ the function $t \mapsto N_t(\omega)$) are cadlag.*

1.4 Poisson processes and Poisson point processes

We say that a measure μ on $\mathbb{R}_+$ is a counting measure if for any $B \in \mathcal{B}(\mathbb{R}_+)$ (i.e. $\mathcal{B}(\mathbb{R}_+)$ is the Borel σ-algebra) we have $\mu(B) \in \mathbb{N}_0$. This is equivalent to say that there is a set $S \subset \mathbb{R}_+$ and for any $x \in S$ there is a point number $n_x \in \mathbb{N}$ such that

$$\mu(\mathrm{d}t) = \sum_{x \in S} n_x \delta_x(\mathrm{d}t)$$

where $\delta_x(\mathrm{d}t)$ is the Dirac delta with unitary mass in $x \in \mathbb{R}_+$, namely

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}.$$

We denote by $\mathbf{N}(\mathbb{R}_+)$ the set of counting measure on $\mathbb{R}_+$.

Definition 1.11. *A random measure $\eta: \Omega \to \mathbf{N}(\mathbb{R}_+)$ is called point process.*

We can build a point process from a Poisson process in the following way

$$\eta^N((a, b]) = N_b - N_a$$

(this is due to the fact that N_t is an *increasing* cadlag function). We call the random measure η^N the Poisson point process of parameter λ (or also the Poisson point process related to the Poisson process N).

Theorem 1.12. *Suppose that η is a point process such that*

1. *for any $B \in \mathcal{B}(\mathbb{R}_+)$ bounded, $\eta(B) \sim \mathrm{Po}(\lambda |B|)$;*
2. *for any $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}_+)$ the random variables $\eta(B_1), \ldots, \eta(B_n)$ are independent.*

Then the stochastic process $N_t = \eta([0, t])$ is a Poisson process. Conversely suppose that η^N is the Poisson point process related to the Poisson process N_t then it satisfies the condition 1 and 2 above.

Remark 1.13. If G is an open set, there are some (at most countable) disjoint intervals $\{I_k = (a_k, b_k)\}_{k \in \mathbb{N}}$ such that $\cup_k I_k = G$.

Furthermore if F is a closed set there are closed intervals $J_k = [a_k, b_k]$ such that $F = \cap_k J_k$.

Remark 1.14. Suppose that $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are sequences of random variables such that $X_n \to X$, $Y_n \to Y$ almost surely and X_n is independent of Y_n then X is independent of Y.

Lemma 1.15. *Suppose that η^N is a Poisson point process then the properties 1, 2 of Theorem 1.12 hold when $B, B_1, \ldots, B_n$ are open or closed sets.*

Proof. Consider first the case where $B_1, \ldots, B_n$ are disjoint intervals $B_k = I_k = (a_k, b_k)$. Then, by continuity of (locally bounded) measure from below, we have that

$$\eta(B_k) = \lim_{n \to +\infty} \eta\left(\left(a_k + \frac{b_k - a_k}{n}, b_k - \frac{b_k - a_k}{n}\right]\right) = \lim_{n \to +\infty} \left(N_{b_k - \frac{b_k - a_k}{n}} - N_{a_k - \frac{b_k - a_k}{n}}\right)$$
$$\sim \mathrm{Po}\left(\lambda\left(\frac{(n-1)}{n}(b_k - a_k)\right)\right).$$

This implies that $\eta(B_k)$ is a Poisson random variable of parameter $\mathrm{Po}(\lambda(b_k-a_k))$. Furthermore since $N_{b_k-\frac{b_k-a_k}{n}}-N_{a_k-\frac{b_k-a_k}{n}}$ is independent of $N_{b_{k'}-\frac{b_{k'}-a_{k'}}{n}}-N_{a_{k'}-\frac{b_{k'}-a_{k'}}{n}}$ (for $k\neq k'$) then $\eta(B_k)$ is independent of $\eta(B_{k'})$ for $k'\neq k$ being the limit of independent random variables.

We can generalize the previous reasoning to general open sets $B_1=G_1,\ \ldots,\ B_n=G_n$ thanks to Remark 1.13. The proof for closed sets follows similar lines. □

Remark 1.16. Lebesgue measure is outer regular: for any (bounded) Borel set B and for any $\varepsilon>0$ there is a open set $G\supset B$ such that $|G\setminus B|<\varepsilon$.

Furthermore if B is Borel set such that $|B|<+\infty$, for any $\varepsilon>0$ there is a closed set $F\subset B$ such that $|B\setminus F|<\epsilon$.

Lemma 1.17. *Suppose that η^N is a Poisson point process then if $|B|=0$ then $\eta^N(B)=0$ almost surely.*

Proof. If $|B|=0$, by Remark 1.16, there is a sequence $B\subset\ldots\subset G^n\subset\ldots\subset G^1$ of open sets such that $|G^n\setminus B|=|G^n|=\frac{1}{n}$. If $K=\cap_{n\in\mathbb{N}}G^n$ then $G\subset K$ and so $\eta^N(B)\leqslant\eta^N(K)$. On the other hand, by the continuity for above of (locally bounded) measure, we have $\eta^N(K)=\lim_{n\to+\infty}\eta^N(G^n)$. Since $\eta^N(G_n)\sim\mathrm{Po}\left(\frac{\lambda}{n}\right)$ we have $\eta^N(K)\sim\mathrm{Po}(0)$ and so $\eta^N(K)=0$ almost surely. Since $\eta^N(B)\leqslant\eta^N(K)=0$ almost surely the thesis follows. □

Lemma 1.18. *Suppose that η^N is a Poisson point process then 1, 2 of Theorem 1.12 hold.*

Proof. By Remark 1.16, for any bounded Borel set B there is a sequence of closed sets $F^1\subset\ldots\subset F^n\subset\ldots\subset B$ such that $|B\setminus F^n|\leqslant\frac{1}{n}$. If $K=\cup_{n\in\mathbb{N}}F^n$, we have that $\eta^N(K)\sim\lim_{n\to+\infty}\mathrm{Po}(\lambda|F^n|)\sim\mathrm{Po}(\lambda|K|)$. On the other hand $|B\setminus K|=\lim_{n\to+\infty}|B\setminus F^n|=0$, and so $\eta^N(B)=\eta^N(K)+\eta^N(B\setminus K)\sim\mathrm{Po}(\lambda|K|)=\mathrm{Po}(\lambda|B|)$, since $|K|=|B|$ and since $\eta^N(B\setminus K)=0$ almost surely, for Lemma 1.17.

Using that if $F_1,\ldots,F_n$ are closed and disjoint, by Lemma 1.15, $\eta^N(F_1),\ \ldots,\ \eta^N(F_n)$ are independent and Remark 1.16. □

Proof of Theorem 1.12. By Lemma 1.18, what remains to prove is that if η is point process satisfying 1 and 2 in the statement of the theorem then the process $N_t=\eta([0,t])$ is a Poisson process.

Since, by the outer continuity of measures, for any t_0 we have $\lim_{t\to t_0^+}\eta([0,t])=\eta([0,t_0])$ the process N_t is right continuous. Furthermore the process N_t is increasing, since $\eta(A)\leqslant\eta(B)$ whenever $A\subset B$, $\lim_{t\to t_0^-}N_t$ exists, which implies that N_t is a process with cadlag paths.

Furthermore for any $t_1\leqslant\ldots\leqslant t_n\in\mathbb{R}_+$, since $N_t-N_s=\eta((s,t])$ (for $s\leqslant t$), we have that, by the properties 1 and 2, $N_{t_i}-N_{t_{i-1}}\sim\mathrm{Po}(\lambda(t_i-t_{i-1}))$ and they are independent, N_t is a Poisson process. □

1.5 Existence of Poisson process and its properties

In this section we build the Poisson process building the Poisson point process associated.

Remark 1.19. In order to build a Poisson point process on $\mathbb{R}_+$ it is sufficient to construct a Poisson point process on $[0,1]$. Indeed suppose that $\tilde{\eta}^1,\ldots,\tilde{\eta}^n,\ldots,$ are a sequence of Poisson point process on $[0,1]$ with parameter $\lambda>0$ and independent. Then we can defined a point process η on $\mathbb{R}_+$ in the following way: if B is a Borel set of $\mathbb{R}_+$ we have

$$\eta(B):=\sum_{n=1}^{+\infty}\tilde{\eta}^n((B-[n-1])\cap(0,1])$$

where $B-k=\{b-k,b\in B\}$.

Consider $P\sim\mathrm{Po}(\lambda)$ (i.e. a Poisson random variable of parameter λ) and let $X_1,\ldots,X_n,\ldots$ be a sequence of i.i.d random variables uniformly distributed on $[0,1]$ (i.e. $X_k\sim U([0,1])$) and independent of P. We defined the random measure $\tilde{\eta}$ on $[0,1]$ in the following way

$$\tilde{\eta}(\omega,\mathrm{d}t)=\sum_{n=1}^{P(\omega)}\delta_{X_n(\omega)}(\mathrm{d}t)$$

where $\omega \in \Omega$ and the sum is equal to zero when $P(\omega) = 0$.

Theorem 1.20. *Using the previous notations, the random measure $\tilde{\eta}$ is a Poisson point process (i.e., by Theorem 1.12 the process $N_t = \tilde{\eta}([0,t])$ is a Poisson process).*

In order to prove Theorem 1.20 we introduce the following sequence of random variables: let $B_1, \ldots, B_n$ be Borel subsets of $[0,1]$ forming a partition of $[0,1]$, for any $r \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$ we defined the random variables

$$Y_k^r(\omega) = \sum_{\ell=1}^{r} \mathbb{I}_{\{X_\ell \in B_k\}}(\omega).$$

Lemma 1.21. *For any $r \in \mathbb{N}$ the random vector $(Y_1^r, \ldots, Y_n^r)$ is distributed as a multinomial random vector of parameter $(p_1 = |B_1|, \ldots, p_n = |B_n|)$, i.e. for any $r_1, \ldots, r_n$ such that $r_1 + \cdots + r_n = r$ we have*

$$\mathbb{P}(Y_1^r = r_1, \ldots, Y_n^r = r_n) = \frac{r!}{r_1! \cdots r_n!} p_1^{r_1} \cdots p_n^{r_n}.$$

Proof. The variable Y_k^r counts the number of points in the set B_k considering a set $(X_1, \ldots, X_r)$ of i.i.d. random variables. This means that the vector $(Y_1^r, \ldots, Y_n^r)$ is distributed as a multinomial random variable with parameter $(p_1 = \mathbb{P}(X_1 \in B_1), p_2 = \mathbb{P}(X_1 \in B_2), \ldots, p_n = \mathbb{P}(X_1 \in B_n))$. Since X_k are uniform random variables $\mathbb{P}(X_k \in B_\ell) = |B_\ell|$. □

Proof of Theorem 1.20. Fix a Borel partition $B_1, \ldots, B_n$ of $[0,1]$ we want to compute the joint probability of the random variables $(\tilde{\eta}(B_1), \ldots, \tilde{\eta}(B_n))$ and prove that it is given by the law of n independents random variables distributed as Poisson random variables with parameter $\tilde{\eta}(B_k) \sim \mathrm{Po}(\lambda |B_k|)$. Since any list of Borel disjoint subsets of $[0,1]$ can be completed to form a partition of $[0,1]$ by adding the complement of the union the theorem is proved.

Consider $r_1, \ldots, r_n \in \mathbb{N}_0$ and $r = r_1 + \cdots + r_n$, we have that

$$\begin{aligned}
\mathbb{P}(\tilde{\eta}(B_1) = r_1, \ldots, \tilde{\eta}(B_n) = r_n) &= \mathbb{P}(P = r, Y_1^r = r_1, \ldots, Y_n^r = r_n) = \\
&= \mathbb{P}(P = r)\mathbb{P}(Y_1^r = r_1, \ldots, Y_n^r = r_n) \\
&= \frac{e^{-\lambda}\lambda^r}{r!} \frac{r!}{r_1! \cdots r_n!} p_1^{r_1} \cdots p^{r_n} \\
&= \prod_{k=1}^{n} \frac{e^{-\lambda p_k}}{r_k!} (p_k \lambda)^{r_k}.
\end{aligned}$$

Since the law of $\mathbb{P}(\tilde{\eta}(B_1) = r_1, \ldots, \tilde{\eta}(B_n) = r_n)$ is a product of functions depending only on r_k the random variables $\tilde{\eta}(B_1), \ldots, \tilde{\eta}(B_n)$ are independent. Furthermore we have that

$$\mathbb{P}(\tilde{\eta}(B_k)) = \frac{e^{-\lambda p_k}}{r_k!} (p_k \lambda)^{r_k}$$

which implies that $\tilde{\eta}(B_k) \sim \mathrm{Po}(\lambda p_k) = \mathrm{Po}(\lambda |B_k|)$. □

1.6 Simplicity of the Poisson process

We want now to prove a fundamental property of the Poisson process, namely that it is a simple process. If X_t is a cadlag stochastic process we define the following process

$$\Delta X_t = X_t - \lim_{s \to t^-} X_t = X_t - X_{t^-}.$$

Definition 1.22. *Let $M_t: \Omega \to \mathbb{N}_0 \subset \mathbb{R}$, $t \in \mathbb{R}_+$, be a stochastic process taking values in the integer number. We say that M_t is simple if $\sup_{t \in \mathbb{R}_+} |\Delta M_t| \leqslant 1$ almost surely, i.e. the process M_t has jump of at most size 1.*

Theorem 1.23. *Every Poisson process is simple.*

In order to prove Theorem 1.23, we introduce the factorial of a discrete measure: suppose that μ is a discrete measure on $\mathbb{R}_+$ taking values in $\mathbb{N}_0$, i.e. there is an at most countable set $S \subset \mathbb{R}_+$ and a map $n.: S \rightarrow \mathbb{N}$ such that

$$\mu(\mathrm{d}t) = \sum_{x \in S} n_x \delta_x(\mathrm{d}t).$$

A measure of the previous for is simple if $n_x < 2$, namely

$$\mu(\{x\}) \leqslant 1, \quad x \in \mathbb{R}_+. \tag{1.6}$$

Remark 1.24. Let η^N be the Poisson point process associated with the Poisson process N_t, then N_t is a simple process is simple if and only if the measure η^N is simple (in the sense of equation (1.6)) almost surely.

We define the second factorial of μ, denoted by $\mu^{(2)}$, as the measure on $\mathbb{R}_+^2$ for which

$$\mu^{(2)}(\mathrm{d}t_1, \mathrm{d}t_2) = \sum_{x_1, x_2 \in S, x_1 \neq x_2} n_{x_1} n_{x_2} \delta_{(x_1, x_2)}(\mathrm{d}t_1, \mathrm{d}t_2) + \sum_{x \in S} n_x(n_x - 1) \delta_{(x,x)}(\mathrm{d}t_1, \mathrm{d}t_2)$$

Remark 1.25. If $D^{(2)} = \{(x,x), x \in \mathbb{R}_+\} \subset \mathbb{R}_+^2$ is the diagonal of $\mathbb{R}_+^2$ we have that

$$\mu^{(2)}(D^{(2)}) = \sum_{x \in S} n_x(n_x - 1) \geqslant \sum_{x \in S} \mathbb{I}_{\{n_y \geqslant 2\}}(x).$$

This means that μ is simple if and only if

$$\mu^{(2)}(D^{(2)}) = 0.$$

Exercise 1.1. Prove that if a B_1, B_2 are Borel sets we have

$$\mu^{(2)}(B_1 \times B_2) = \mu(B_1)\mu(B_2) - \mu(B_1 \cap B_2).$$

Lemma 1.26. *For any Borel sets $B_1, B_2 \in \mathcal{B}(\mathbb{R}_+)$ we have*

$$\mathbb{E}[\eta^{N,(2)}(B_1 \times B_2)] = \lambda^2 |B_1||B_2|.$$

Proof. By Exercise 1.1 we have that

$$\eta^{N,(2)}(B_1 \times B_2) = \eta^N(B_1)\eta^N(B_2) - \eta^N(B_1 \cap B_2).$$

Thus we have

$$\begin{aligned}
\mathbb{E}[\eta^{N,(2)}(B_1 \times B_2)] &= \mathbb{E}[\eta^N(B_1)\eta^N(B_2)] - \mathbb{E}[\eta^N(B_1 \cap B_2)] \\
&= \mathbb{E}[\eta^N(B_1 \backslash (B_1 \cap B_2))\eta^N(B_2 \backslash (B_1 \cap B_2))] + \\
&\quad + \mathbb{E}[\eta^N(B_1 \backslash (B_1 \cap B_2))\eta^N(B_1 \cap B_2)] + \\
&\quad + \mathbb{E}[\eta^N(B_1 \cap B_2)\eta^N(B_2 \backslash (B_1 \cap B_2))] + \mathbb{E}[(\eta^N(B_1 \cap B_2))^2] \\
&\quad - \mathbb{E}[\eta^N(B_1 \cap B_2)] \\
&= \lambda^2 |B_1 \backslash (B_1 \cap B_2)||B_2 \backslash (B_1 \cap B_2)| + \lambda^2 |B_1 \backslash (B_1 \cap B_2)|\,|B_1 \cap B_2| + \\
&\quad + \lambda^2 |B_1 \cap B_2||B_2 \backslash (B_1 \cap B_2)| + \lambda^2 |B_1 \cap B_2|^2 + \lambda |B_1 \cap B_2| - \lambda |B_1 - B_2| \\
&= \lambda^2 |B_1|\,|B_2|,
\end{aligned}$$

where we used that $\eta^N(L_1)$ is independent of $\eta^N(L_2)$ when $L_1 \cap L_2 = \emptyset$, and $\mathbb{E}[\eta^N(L_1)] = \lambda |L_1|$ and $\mathbb{E}[(\eta^N(L_1))^2] = \lambda^2 |L_1|^2 + \lambda |L_1|$. □

Proof of Theorem 1.23. By Remark 1.24 and Remark 1.25, the process N_t is simple if and only if $\eta^{N,(2)}(D^{(2)}) = 0$ almost surely. Since $\eta^{N,(2)}(D^{(2)}) \geqslant 0$ is enough to prove that $\mathbb{E}[\eta^{N,(2)}(D^{(2)})] = 0$.

Then, by Lemma 1.26, we have that

$$\begin{aligned}\mathbb{E}[\eta^{N,(2)}(D^2)] &= \sum_{k=0}^{+\infty} \mathbb{E}[\eta^{N,(2)}(D^2 \cap [k,k+1)^2)]\\ &= \sum_{k=0}^{+\infty} \left(\lim_{n\to+\infty} \sum_{r=1}^{n} \mathbb{E}\left[\eta^{N,(2)}\left(D^2 \cap \left[k+\frac{r-1}{n}, k+\frac{r}{n}\right)\right)\right]\right)\\ &= \sum_{k=0}^{+\infty} \left(\lim_{n\to+\infty} \sum_{r=1}^{n} \lambda^2 \left|\left[k+\frac{r-1}{n}, k+\frac{r}{n}\right)\right|^2\right)\\ &= \sum_{k=0}^{+\infty} \left(\lim_{n\to+\infty} \frac{\lambda^2}{n}\right) = 0.\end{aligned}$$

□

1.7 Levy processes

Definition 1.27. *Let X_t be a stochastic process, we say that X_t is a Levy process if*

1. $X_0 = 0$,
2. *the process X_t has independent and homogeneous increments, i.e. for any $t_1 \leqslant \cdots \leqslant t_n$ we have $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent and $X_{t_i} - X_{t_{i-1}} \sim X_{t_i - t_{i-1}}$ (i.e. $X_{t_i} - X_{t_{i-1}}$ has the same law of $X_{t_i - t_{i-1}}$),*
3. *the process X_t has cadlag paths, for each $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is cadlag.*

Two important examples of Levy processes are Brownian motion and Poisson process.

From the definition we can deduce that the law of the process X_t to a fixed time $t > 0$ cannot be a generic law but it must be *infinite divisible*.

Definition 1.28. *We say that a probability measure μ on $\mathbb{R}$ is infinite divisible if for any $n \in \mathbb{N}$ there is a probability measure $\mu^{\frac{1}{n}}$ such that*

$$\mu = \underbrace{\mu^{\frac{1}{n}} * \mu^{\frac{1}{n}} * \cdots * \mu^{\frac{1}{n}}}_{n \quad \text{times}}.$$

Exercise 1.2. Suppose that Y_1 and Y_2 are two *independent* random variables with probability law μ_1 and μ_2 respectively. Prove that the of $Y_1 + Y_2$ is given by $\mu_1 * \mu_2$ (i.e. for any Borel set $B \in \mathcal{B}(\mathbb{R})$ we have $\mu_1 * \mu_2(B) = \int_{\mathbb{R}} \mu_1(B - x)\mu_2(\mathrm{d}x)$.

Theorem 1.29. *Let X_t be a Levy process, then, for any $t \in \mathbb{R}_+$, the law of X_t is infinite divisible.*

Proof. Obviously since $X_0 = 0$, the law of X_0 is infinite divisible. Then, consider $t > 0$. For any $n \in \mathbb{N}$ we have that

$$X_t = \sum_{k=0}^{n-1} X_{\frac{t(k+1)}{n}} - X_{\frac{tk}{n}}.$$

By definition of Levy process $X_{\frac{t(k+1)}{n}} - X_{\frac{tk}{n}}$ are i.i.d random variables. Then, by Exercise 1.2, if $X_t \sim \mu$ and $X_{\frac{t(k+1)}{n}} - X_{\frac{tk}{n}} \sim X_{\frac{t}{n}} \sim \mu^{\frac{1}{n}}$ we have $\mu = \mu^{\frac{1}{n}} * \cdots * \mu^{\frac{1}{n}}$. □

There is a converse of Theorem 1.29 that we state without proof.

Theorem 1.30. *Let μ be an infinite divisible probability measure, then there is a Levy process such that $X_1 \sim \mu$.*

We want to give an important class of exmples of Levy processes other than Brownian motion and Poisson process.

Definition 1.31. *Consider a Poisson process N_t with a paramter $\lambda>0$ and let $Y_1,\ldots,Y_n,\ldots$ be a sequence of i.i.d. random variables indepedent of the process N. We call X_t a compound Poisson process with paramter λ and jump* $\mathrm{Law}(Y_i)$ *the process*

$$X_t(\omega)=\sum_{k=1}^{N_t(\omega)} Y_k(\omega)$$

where the sum is equal to zero when $N_t(\omega)=0$.

Theorem 1.32. *Compound Poisson processes are Levy process.*

Proof. Since $N_0=0$, $X_0=0$.

In order to prove the theorem it is useful to introduce the process

$$Z_t=\sum_{k=1}^{\lfloor t\rfloor} Y_k$$

where $Z_t=0$ for $t<1$. The process Z_t is a cadlag process (i.e. it is a process with cadlag paths), furthermore we have that

$$X_t(\omega)=Z_{N_t(\omega)}(\omega),$$

i.e. for any $\omega\in\Omega$ the map $t\mapsto X_t(\omega)$ is the composition of the cadlag function $t\mapsto Z_t(\omega)$ and the cadlag increasing function $t\mapsto N_t(\omega)$. Since the composition of a cadlag function with a increasing cadlag function is cadlag, the function $t\mapsto X_t(\omega)$ is cadlag.

What reamian to prove is that X_t has independent and homogeneous increments. We prove the statement for $t_1\leqslant t_2\leqslant t_3$, since the general case can be proven in a similar way. We write

$$K_1=X_{t_2}-X_{t_1},\quad K_2=X_{t_3}-X_{t_2},\quad K_1'=X_{t_2-t_1},\quad K_2'=X_{t_3-t_2}$$

and we want to prove that

$$\varphi_{(K_1,K_2)}(u,v)=\mathbb{E}[\exp(iuK_1+ivK_2)]=\varphi_{K_1'}(u)\varphi_{K_2'}(v)=\mathbb{E}[\exp(iuK_1')]\mathbb{E}[\exp(ivK_2')] \tag{1.7}$$

where $\varphi_{(K_1,K_2)}$, $\varphi_{K_1'}$, $\varphi_{K_2'}$ are the characteristic functions of (K_1,K_2), K_1', K_2' respectively. Equation (1.7) will imply that $X_{t_3}-X_{t_2}$ is independent of $X_{t_2}-X_{t_1}$ (since two random variables are independent if and only if their joint characteristic function is the product of their marginal charateristic functions) and their are distributed as $X_{t_3}-X_{t_2}\sim X_{t_3-t_2}$ and $X_{t_2}-X_{t_1}\sim X_{t_2-t_1}$.

By definition of Poisson process we have that $N_{t_3}-N_{t_2}$, $N_{t_2}-N_{t_1}$ and N_{t_1} are independent we have

$$\begin{aligned}
\varphi_{(K_1,K_2)}(u,v) &= \mathbb{E}[\mathbb{E}[e^{iuK_1+ivK_2}|N_{t_1},N_{t_2}-N_{t_1},N_{t_3}-N_{t_2}]]\\
&= \sum_{k_1,k_2,k_3\in\mathbb{N}} \mathbb{P}(N_{t_3}-N_{t_2}=k_1,N_{t_2}-N_{t_1}=k_2,N_{t_1}=k_3)\\
&\quad \mathbb{E}[e^{iuK_1+ivK_2}|N_{t_3}-N_{t_2}=k_1,N_{t_2}-N_{t_3}=k_2,N_{t_1}=k_3]\\
&= \sum_{k_1,k_2,k_3\in\mathbb{N}} \mathbb{P}(N_{t_3}-N_{t_2}=k_1)\mathbb{P}(N_{t_2}-N_{t_3}=k_2)\mathbb{P}(N_{t_1}=k_3)\\
&\quad \mathbb{E}\Big[e^{iu(\sum_{j=k_3+k_2+1}^{k_1+k_2+k_3}Y_j)+iv(\sum_{j=k_3+1}^{k_1+k_2}Y_j)}\Big]\\
&= \sum_{k_1,k_2,k_3\in\mathbb{N}} \mathbb{P}(N_{t_3}-N_{t_2}=k_1)\mathbb{P}(N_{t_2}-N_{t_3}=k_2)\mathbb{P}(N_{t_1}=k_3)\\
&\quad \mathbb{E}\Big[e^{iu(\sum_{j=k_3+k_2+1}^{k_1+k_2+k_3}Y_j)}\Big]\mathbb{E}\Big[e^{iv(\sum_{j=k_3+1}^{k_1+k_2}Y_j)}\Big]=\\
&= \sum_{k_1,k_2\in\mathbb{N}} \mathbb{P}(N_{t_3-t_2}=k_1)\mathbb{P}(N_{t_2-t_1}=k_2)\mathbb{E}\Big[e^{iu(\sum_{j=1}^{k_1}Y_j)}\Big]\\
&\quad \mathbb{E}\Big[e^{iv(\sum_{j=1}^{k_2}Y_j)}\Big]\\
&= \mathbb{E}[e^{iuK_1'}]\mathbb{E}[e^{ivK_2'}]=\varphi_{K_1'}(u)\varphi_{K_2'}(v).
\end{aligned}$$

□

Chapter 2

Filtrations, martingales and stopping times

2.1 Some definitions

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1. *We say that $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is a filtration if $\mathcal{F}_t \subset \mathcal{F}$ are σ-algebras such that for any $s \leqslant t$ we have $\mathcal{F}_s \subset \mathcal{F}_t$.*

We write

$$\mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \in \mathbb{R}_+).$$

Definition 2.2. *A stochastic process M_t is called adapted to the filtration $\mathcal{F}_t$ if for any $s \in \mathbb{R}_+$ M_t is $\mathcal{F}_t$ measurable.*

A stochastic process M_t is called a cadlag process if for any $\omega \in \Omega$ the map $t \mapsto M_t(\omega)$ is cadlag.

Definition 2.3. *Let X be a stochastic process we call*

$$\mathcal{F}_t^X = \sigma(X_s, s \leqslant t)$$

the natural filtration of (or the natural filtration generated by) X.

Remark 2.4. A stochastic process is obviously adapted with respect to its natural filtration.

Definition 2.5. *Let M_t be an adapted stochastic process we say that $\mathbb{E}[|M_t|] < +\infty$ we say that:*

- *M_t is a $(\mathcal{F}_t)$-martingale if $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$;*
- *M_t is a $(\mathcal{F}_t)$-supermartingale if $\mathbb{E}[M_t|\mathcal{F}_s] \leqslant M_s$;*
- *M_t is a $(\mathcal{F}_t)$-submartingale if $\mathbb{E}[M_t|\mathcal{F}_s] \geqslant M_s$.*

We say that M_t is a cadlag (super/sub)martingale if M_t is cadlag and it is also a (super/sub)martingale.

We denote by

$$\tilde{\mathcal{N}} = \{A \in \mathcal{F}, \mathbb{P}(A) = 0\},$$

$$\mathcal{N} = \sigma\{A \in \mathcal{F}, \mathbb{P}(A) = 0\}.$$

Definition 2.6. *Let $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ be a filtration we define the completion $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ of filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ as*

$$\mathcal{G}_t = \sigma(\mathcal{F}_t, \mathcal{N}) = \{A \in \mathcal{F}, \exists B \in \mathcal{F}_t \, \text{such that} \, \mathbb{P}(A \Delta B) = 0\}$$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$. If a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ coincide with its completion we say that $\mathcal{F}_t$ is complete.

If $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is a filtration we define

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_t.$$

Remark 2.7. In general $\mathcal{F}_t \neq \mathcal{F}_{t+}$. Indeed let $\mathcal{F}_t^B$ be the natural filtration of a Brownian motion and define

$$A_t = \{\omega \in \Omega, B_t(\omega) \text{ is right differentiable at } t\} = \left\{\omega \in \Omega, \lim_{n \to +\infty} n\left(B_{t+\frac{1}{n}}(\omega) - B_t(\omega)\right) \text{exists}\right\}.$$

Obviously $A_t \in \mathcal{F}_{t+}^B$ but $A_t \notin \mathcal{F}_t^B$ since it is not possible to know if a function is right differentiable if we have only information of a function from the left.

Definition 2.8. *Let $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ be a filtration the right-continuous completion of $\mathcal{F}_t$ the filtration $\{\mathcal{H}_t\}_{t \in \mathbb{R}_+}$ the filtration defined as*

$$\mathcal{H}_t = \sigma(\mathcal{F}_{t+}, \mathcal{N}).$$

If $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ coincides with its right-continuous completion we say that it is complete and right-continuous.

If for any $t \in \mathbb{R}_+$ we have $\mathcal{F}_t = \mathcal{F}_{t+}$ we say that $\mathcal{F}_t$ is right continuous.

Definition 2.9. *We say that the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ satisfies the usual condition if it is complete and it is right-continuous.*

2.2 Some theorems on filtrations

Theorem 2.10. *Let M_t be a (super/sub)martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$, then it is also a (super/sub)martingale with respect to $\{\mathcal{G}_t\}_{t \in \mathbb{R}_+}$ (where $\{\mathcal{G}_t\}_{t \in \mathbb{R}_+}$ is the completion of $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$).*

Proof. Exercise. □

Theorem 2.11. *Let M_t be a right-continuous (super/sub)martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$, then it is also a (super/sub)martingale with respect to $\{\mathcal{H}_t\}_{t \in \mathbb{R}_+}$ (where $\{\mathcal{H}_t\}_{t \in \mathbb{R}_+}$ is the right-continuous completion of $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$).*

Proof. The proof can be found in [7] Chapter 1 Section 1 (the proof not required at the exam) □

Theorem 2.12. *Let B_t be a Brownian motion and let $\mathcal{G}_t^B$ its complete natural filtration then $\mathcal{G}_t^B$ is right continuous (i.e. with the previous notations $\mathcal{H}_t^B = \mathcal{G}_t^B$).*

Lemma 2.13. *Let B_t be a Brownian motion and $\alpha \in \mathbb{C}$ then*

$$F^\alpha(B_t, t) = \exp\left(\alpha B_t - \frac{\alpha^2 t}{2}\right)$$

is a martingale with respect to its natural filtration $\mathcal{F}_t^B$ (and so with respect to the completed natural filtration $\mathcal{G}_t^B$).

Proof. For any $s \leqslant t$ we have that

$$\begin{aligned}
\mathbb{E}[F(B_t, t)|\mathcal{F}_s^B] &= \mathbb{E}\left[\exp\left(\alpha B_t - \frac{\alpha^2 t}{2}\right)\middle|\mathcal{F}_t\right] \\
&= \exp\left(\alpha B_s - \frac{\alpha^2 t}{2}\right)\mathbb{E}[\exp(\alpha B_t - \alpha B_s)|\mathcal{F}_t] \\
&= \exp\left(\alpha B_s - \frac{\alpha^2 t}{2}\right)\mathbb{E}[\exp(\alpha B_t - \alpha B_s)] \\
&= \exp\left(\alpha B_s - \frac{\alpha^2 t}{2}\right)\exp\left(\frac{\alpha^2 (t-s)}{2}\right) = F^\alpha(B_s, s)
\end{aligned}$$

where we use that $B_t - B_s$ is independent of $\mathcal{F}_s^B$. □

Proof of Theorem 2.12. For simplicity we write $\mathcal{G}_t = \mathcal{G}_t^B$. The statement is equivalent to prove that

$$\mathcal{G}_t = \bigcap_{n \in \mathbb{N}} \mathcal{G}_{t+\frac{1}{n}}$$

for each $t > 0$, so all the limits $s \downarrow t$ with countable limits $t + \frac{1}{n} \to t$ for $n \to +\infty$.

Fix $t, s_1, \ldots, s_m \geqslant 0$ and $u_1, \ldots, u_m \in \mathbb{R}$ we want to prove that

$$\mathbb{E}\left[\exp\left(i\left(\sum_{k=1}^{m} u_k B_{s_k}\right)\right)\middle|\mathcal{G}_t\right] = \mathbb{E}\left[\exp\left(i\left(\sum_{k=1}^{m} u_k B_{s_k}\right)\right)\middle|\mathcal{G}_{t+}\right]. \tag{2.1}$$

The equality is obviously true when $s_1, \ldots, s_m \leqslant t$, so we have to prove only the case where $\min(s_k) > t$.

We prove explicitly only the case $m = 2$ and $s_2 > s_1 > t$, the general case can be proved in a similar way. We have, by Doob martingale convergence (using the fact $\mathbb{E}\left[\exp(i(u_1 B_{s_1} + u_2 B_{s_2}))\middle|\mathcal{G}_{t+\frac{1}{n}}\right]$ is a uniformly integrable martingale (since it is bounded))

$$\begin{aligned}
\mathbb{E}[\exp(i(u_1 B_{s_1} + u_2 B_{s_2}))|\mathcal{G}_{t+}] &= \lim_{n \to +\infty} \mathbb{E}\left[\exp(i(u_1 B_{s_1} + u_2 B_{s_2}))\middle|\mathcal{G}_{t+\frac{1}{n}}\right] \\
&= \lim_{n \to +\infty} \exp\left(-\frac{u_2^2 s_2}{2}\right)\mathbb{E}\left[\mathbb{E}[\exp(i(u_1 B_{s_1}))F^{iu_2}(B_{s_2}, s_2)|\mathcal{G}_{s_1}]\middle|\mathcal{G}_{t+\frac{1}{n}}\right] \\
&= \lim_{n \to +\infty} \exp\left(-\frac{u_2^2(s_2 - s_1)}{2}\right)\mathbb{E}\left[\exp(i(u_1 + u_2)B_{s_1})\middle|\mathcal{G}_{t+\frac{1}{n}}\right] \\
&= \lim_{n \to +\infty} \exp\left(-\frac{u_2^2(s_2 - s_1)}{2} - \frac{(u_1 + u_2)^2}{2}s_1\right) \\
&\quad \mathbb{E}\left[F^{i(u_1+u_2)}(B_{s_1}, s_1), \middle|\mathcal{G}_{t+\frac{1}{n}}\right] \\
&= \lim_{n \to +\infty} \exp\left(-\frac{u_2^2(s_2 - s_1)}{2} - \frac{(u_1 + u_2)^2}{2}s_1\right) \\
&\quad F^{i(u_1+u_2)}\left(B_{t+\frac{1}{n}}, t + \frac{1}{n}\right) \\
&= \exp\left(-\frac{u_2^2(s_2 - s_1)}{2} - \frac{(u_1 + u_2)^2}{2}s_1\right)F^{i(u_1+u_2)}(B_t, t) \\
&= \mathbb{E}[\exp(i(u_1 B_{s_1} + u_2 B_{s_2}))|\mathcal{G}_t],
\end{aligned}$$

where in the last step we used that B_t is continuous. Since the measure is uniquely determined by the characteristic function, a consequence of the equality (2.1) for any Borel set $A \subset \mathbb{R}^m$ we have that the conditional probability

$$\mathbb{P}((B_{s_1}, \ldots, B_{s_m}) \in A|\mathcal{G}_{t+}) = \mathbb{P}((B_{s_1}, \ldots, B_{s_m}) \in A|\mathcal{G}_t)$$

almost surely. Furthermore since the sets of the form

$$\{(B_{s_1}, \ldots, B_{s_m}) \in A\}$$

where m, s_k, A vary in $m \in \mathbb{N}$, $s_1, \ldots, s_m \in \mathbb{R}_+$ and $A \in \mathcal{B}(\mathbb{R}^m)$, generates the σ-algebra $\mathcal{G}_\infty$ we have that for any $C \in \mathcal{G}_\infty$

$$\mathbb{P}(C|\mathcal{G}_{t+}) = \mathbb{P}(C|\mathcal{G}_t)$$

almost surely. In particular if $C \in \mathcal{G}_{t+}$ we have

$$\mathbb{I}_C = \mathbb{P}(C \mid \mathcal{G}_{t+}) = \mathbb{P}(C|\mathcal{G}_t) = \mathbb{E}[\mathbb{I}_C|\mathcal{G}_t]$$

almost surely. Since $\mathcal{G}_t$ is complete the previous equality proves that $C \in \mathcal{G}_t$, and so $\mathcal{G}_{t+} \subset \mathcal{G}_t$. □

We conclude this section with a definition and a theorem (whose prove is not required for the exam).

Definition 2.14. *Let X_t and Y_t be two stochastic process we say that*

- *the process Y is a modification of X if for any $t \geqslant 0$ we have $\mathbb{P}(X_t = Y_t) = 1$;*

- *the processes Y and X are indistinguishable if* $\mathbb{P}(X_t = Y_t, t \in \mathbb{R}_+) = 1$.

Remark 2.15. Suppose that X, Y are two right-continuous processes, then the definition of indistiguishability and modification coincide.

Theorem 2.16. *Let $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ be a filtration and M_t be a (sub)martingale with respect to $\mathcal{H}_t$. Suppose also that $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ is complete and right-continuous (i.e. $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ satisfies the usual condition), that $\mathcal{H}_0$ contains all the null-sets, and the map $t \to \mathbb{E}[M_t]$ is right-continuous then there is a modification $\tilde{M}_t$ of M_t which is a cadlag (sub)martingale with respect to $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$.*

2.3 Stopping times

We fix a general filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ (i.e. here we do not require that $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is complete or right-continuous if not state otherwise).

Definition 2.17. *Let $T: \Omega \to \mathbb{R}_+$ be a $\mathcal{F}$ measurable random variable. We say that T is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ if for any $t \in \mathbb{R}_+$ we have*

$$\{T \leqslant t\} \in \mathcal{F}_t.$$

Remark 2.18. We say that $T: \Omega \to \mathbb{R}_+$ is a $\mathcal{F}_{t+}$ stopping time if for any $t \in \mathbb{R}_+$ we have

$$\{T \leqslant t\} \in \mathcal{F}_{t+}.$$

Since

$$\{T < t\} = \bigcup_{n\in\mathbb{N}} \left\{T \leqslant t - \frac{1}{n}\right\} \in \sigma\left(\mathcal{F}_{\left(t-\frac{1}{n}\right)_+}, n \in \mathbb{N}\right) \subset \mathcal{F}_t$$

we have that T is a $\mathcal{F}_{t+}$ stopping time if and only if for any $t \in \mathbb{R}_+$

$$\{T < t\} \in \mathcal{F}_t.$$

We want to do some examples of stopping times which will be useful in the following. Let $B \in \mathcal{B}(\mathbb{R})$ be a Borel set we define

$$\tau_B = \inf\{t \geqslant 0, X_t \in B\},$$

$$\tilde{\tau}_B = \inf\{t > 0, X_t \in B\}.$$

The random variable τ_B is called *first entrance time* of the set B, and $\tilde{\tau}_B$ is called *first hitting time* of the set B.

If X has left limit we define

$$\sigma_B = \inf\{t \geqslant 0, X_t \in B \text{ or } X_{t-} \in B\},$$

$$\tilde{\sigma}_B = \inf\{t > 0, X_t \in B \text{ or } X_{t-} \in B\},$$

Obviously if X_t is continuous we have $\sigma_B = \tau_B$ and $\tilde{\sigma}_B = \sigma_B$.

Proposition 2.19. *Suppose that X_t is a cadlag or a left continuous process and $B = G$ is an open set then τ_G and $\tilde{\tau}_G$ are $\mathcal{F}_{t+}$ stopping times. In particular if $\mathcal{F}_t$ is right continuous then τ_G and $\tilde{\tau}_G$ are $(\mathcal{F}_t)$stopping times.*

Proof. Since the paths $t \mapsto X_t(\omega)$ are right-continuous we have that $\tau_G(\omega), \tilde{\tau}_G(\omega) < t$ if and only if there is $s \in [0, t)$ (or $s \in (0, t)$ in the case of $\tilde{\tau}_G$) such that $X_s(\omega) \in G$. Since X is right continuous and G is open this is equivalent to say that there is a $q \in [0, t) \cap \mathbb{Q}$ (or $(0, t) \cap \mathbb{Q}$ for $\tilde{\tau}_G$) such that $X_q(\omega) \in G$. This means that

$$\{\tau_G < t\} = \bigcup_{q\in[0,t)\cap\mathbb{Q}} \{X_q \in G\} \in \sigma(X_s, s \in [0, t)) \subset \mathcal{F}_t$$

$$\{\tilde{\tau}_G < t\} = \bigcup_{q\in(0,t)\cap\mathbb{Q}} \{X_q \in G\} \in \sigma(X_s, s \in [0, t)) \subset \mathcal{F}_t.$$

By Remark 2.18 the thesis is proved. □

Proposition 2.20. *Suppose that $B = F$ is a closed subset of $\mathbb{R}$ and that X_t is cadlag, then σ_F are $(\mathcal{F}_t)$stopping times.*

Proof. We start by proving that

$$\{\sigma_F \leqslant t\} = \{X_0 \in F\} \cup \{X_s \in F \text{ and } X_{s-} \in F, \text{for some } s \in (0,t]\}$$

It is clear that the event on the left is contained in the event on the right. To prove the opposite inclusion suppose that $\sigma_F \leqslant t$. If the event on the right does not happen for any $k \in \mathbb{N}$ there is $t < u_k < t + \frac{1}{k}$ such that or X_{u_k} or X_{u_k-} are in F. Since $u_k \to t$, as $k \to +\infty$, both X_{u_k}, X_{u_k-} converge to X_t as $u_k \to t$ and since F is closed $X_t \in F$. This proves the equality.

Let

$$F_n = \left\{ y \in \mathbb{R}, \text{there is } x \in F \text{ such that } |x - y| < \frac{1}{n} \right\},$$

i.e. F_n is the $\frac{1}{n}$ neighborhood of F. We now want to prove that

$$\{X_0 \in F\} \cup \{X_s \in F \text{ and } X_{s-} \in F, \text{for some } s \in (0,t]\} = \bigcap_{n \in \mathbb{N}} \bigcup_{q \in [0,t] \cap \mathbb{Q}} \{X_q \in F_n\}$$

Indeed, suppose that $X_s \in F$ for some $s \in [0,t]$ then we can find a sequence $q_j \in [0,t] \cap \mathbb{Q}$ (or $(0,t] \cap \mathbb{Q}$) such that $X(q_j) \in H_n$ for j big enough. This prove that the left hand side of the previous equality is contained in the right hand side.

Conversely suppose that there is a sequence $q_n \in [0,t] \cap \mathbb{Q}$ such that $X_{q_n} \in F_n$ for each $n \in \mathbb{N}$. Since $[0,t]$ is closed there is a increasing or decreasing subsequence q_{n_j} converging to some $s \in [0,t]$. This means that X_{q_j} converges to X_s (if q_j is decreasing) or to X_{s-} (if the sequence is decreasing). This means that or X_s or X_{s-} must belong to $F = \bigcap_{n \in \mathbb{N}} F_n$. □

Proposition 2.21. *Suppose that $B = F$ is a closed subset of $\mathbb{R}$ and that X_t is cadlag, then $\tilde{\sigma}_F$ are $(\mathcal{F}_t)$stopping times.*

Proof. We note that

$$\{\tilde{\sigma}_F \leqslant t\}^c = (\{\sigma_F \leqslant t\})^c \cup (\{\tilde{\tau}_{F^c} = 0\} \cap \{\sigma_F \leqslant t\})$$

Indeed the only possibility that $\tilde{\sigma}_F$ is bigger then t but $\sigma_F \leqslant t$ is that $X_0 \in F$ but $X_s \notin F$ for each $s \in (0,t]$ (this is true because F is closed and X is right-continuous and so if $X_k \in F$ in a neighborhood of $t+$ then $X_t \in F$) which is equivalent to say that $\{\tilde{\tau}_{F^c} = 0\}$ and $\sigma_F \leqslant t$. By the previous propositions we have that $\{\sigma_F \leqslant t\} \in \mathcal{F}_t \subset \mathcal{F}_{t+}$ and $\{\tilde{\tau}_{F^c} = 0\} \in \mathcal{F}_{0+}$. Since $\mathcal{F}_{t+}$ is a σ-algebra this implies that $\{\tilde{\sigma}_F \leqslant t\}^c \in \mathcal{F}_{t+}$, which means that $\{\tilde{\sigma}_F \leqslant t\} \in \mathcal{F}_{t+}$. □

Corollary 2.22. *If the process X_s is continuous and F is closed τ_F is a $(\mathcal{F}_t)$stopping time and $\tilde{\tau}_F$ a $\mathcal{F}_{t+}$ stopping time.*

We want to study the composition of a process X_t and a stopping time. First we introduce the following definition.

Definition 2.23. *If T is a stopping time we defined the σ-algebra $\mathcal{F}_T$ as $\Lambda \in \mathcal{F}_T$ if and only if for any $t \geqslant 0$ we have*

$$\Lambda \cap \{T \leqslant t\} \in \mathcal{F}_t.$$

Proposition 2.24. *Let T, S two stopping times then*

1. *if $S \leqslant T$ then $\mathcal{F}_S \subset \mathcal{F}_T$;*
2. *$\mathcal{F}_{T \wedge S} = \mathcal{F}_T \cap \mathcal{F}_S$;*
3. *if $F \in \mathcal{F}_{T \vee S}$ then $F \cap \{S \leqslant T\} \in \mathcal{F}_T$;*
4. *$\mathcal{F}_{T \vee S} = \sigma(\mathcal{F}_T, \mathcal{F}_S)$.*

Proof. Exercise. □

Even if X_t is a process adapted with respect to the filtration $\mathcal{F}_t$ is not true (in general) that X_T is a $\mathcal{F}_T$ measurable random variable. We need then the following definition.

Definition 2.25. *A process X_t is said to be progressive with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ if for any $t\in\mathbb{R}_+$ the map $X_\cdot(\cdot)\colon [0,t]\times\Omega\to\mathbb{R}$ is $\mathcal{B}([0,t])\otimes\mathcal{F}_t$ measurable.*

Theorem 2.26. *Suppose that X is progressive and T is a stopping time then X_T is $\mathcal{F}_T$ measurable on the event $\{T<+\infty\}$.*

Proof. We claim that for any $t\in\mathbb{R}_+$ the map

$$\omega\mapsto X_{T(\omega)\wedge t}(\omega)$$

is $\mathcal{F}_t$ measurable. Obviously $T\wedge t$ is a stopping time and $T\wedge t$ is $\mathcal{F}_t$ measurable. This means (by definition of stopping time as measurable map taking values in $\mathbb{R}_+$) that the map

$$\omega\mapsto (T(\omega)\wedge t,\omega)\in[0,t]\times\Omega$$

is a measurable map form measure space $(\Omega,\mathcal{F}_t)$ to the measure space $([0,t]\times\Omega,\mathcal{B}([0,t])\otimes\mathcal{F}_t)$. By definition of progressive the map

$$(s,\omega)\in[0,t]\times\Omega\mapsto X_s(\omega)\in\mathbb{R}$$

is measurable from the measure space $([0,t]\times\Omega,\mathcal{B}([0,t])\otimes\mathcal{F}_t)$ into $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. This implies that the composition of the two maps $\omega\mapsto(T(\omega)\wedge t,\omega)$ with $(s,\omega)\mapsto X_s(\omega)$ (which is $\omega\mapsto X_{T(\omega)\wedge t}(\omega)$) is a measurable map from $(\Omega,\mathcal{F}_t)$ into $\mathbb{R}$. The previous statement means that

$$\{X_{T\wedge t}\in B\}\in\mathcal{F}_t$$

for any Borel set $B\in\mathcal{B}(\mathbb{R})$, and so

$$\{X_T\in B, T<+\infty\}\cap\{T\leqslant t\}=\{X_{T\wedge t}\in B\}\cap\{T\leqslant t\}\in\mathcal{F}_t.$$

This show that $\{X_T\in B,T<+\infty\}\in\mathcal{F}_T$ proving the claim. □

Proposition 2.27. *Let X_t be an adapted cadlag process with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ then it is also progressive.*

Proof. Fix $\tau>0$ define on $[0,\tau]\times\Omega$ the function

$$X_n(t,\omega)=X_0(\omega)\mathbb{I}_{\{0\}}(t)+\sum_{k=0}^{2^n-1}X_{\frac{(k+1)\tau}{2^n}}(\omega)\mathbb{I}_{\left(\frac{k\tau}{2^n},\frac{(k+1)\tau}{2^n}\right]}(t).$$

The function X_n is a sum of products of $\mathcal{B}([0,\tau])\otimes\mathcal{F}_\tau$ measurable functions and thus it is $\mathcal{B}([0,\tau])\otimes\mathcal{F}_\tau$. By right-continuity of X_t we have that, for any $(t,\omega)\in[0,\tau]\times\Omega$, $X_n(t,\omega)\to X_t(\omega)$ as $n\to+\infty$, thus X_t is also $\mathcal{B}([0,\tau])\otimes\mathcal{F}_\tau$ when restricted to $[0,\tau]$. □

2.4 Doob's optional sampling theorem

The aim of this section is to prove the following results.

Lemma 2.28. *Let T,S be $\mathcal{F}_t$ stopping times taking values in a finite set $t_1\leqslant t_2\leqslant\cdots\leqslant t_m\leqslant+\infty$. If X_t is a submartingale then*

$$\mathbb{E}[X_T|\mathcal{F}_S]\geqslant X_S$$

almost surely.

Proof. The proof can be found in Chapter 1 Lemma 1.45 of [1]. □

Theorem 2.29. *Let M be a cadlag submartingale with respect to $\mathcal{F}_t$ and let T,S be $\mathcal{F}_t$-stopping times then for any $\tau>0$*

$$\mathbb{E}[X_{T\wedge\tau}|\mathcal{F}_S]\geqslant X_{T\wedge S\wedge\tau},$$

almost surely. If, in addition,

1. *T is almost surely finite,*
2. *$\mathbb{E}[|X_T|] < +\infty$,*
3. *$\lim_{\tau\to+\infty}\mathbb{E}[|X_\tau|\mathbb{I}_{T>\tau}] = 0$*

then

$$\mathbb{E}[X_T|\mathcal{F}_S] \geqslant X_{T\wedge S},$$

almost surely. Finally an analogous theorem holds for cadlag supermartingales and martingales.

Proof. The proof uses Lemma 2.28 and it can be found in Chpater 1 Theorem 1.43 of [1]. □

2.5 Martingale inequalities

Lemma 2.30. *Let M be a submartingale, fix a $0 < \tau < +\infty$ and let $H \subset [0,\tau]$ be a finite set. Then for any $r > 0$ we have*

$$\mathbb{P}\Big(\max_{t\in H} M_t \geqslant r\Big) \leqslant \frac{\mathbb{E}[M_\tau^+]}{r}$$

and

$$\mathbb{P}\Big(\min_{t\in H} M_t \leqslant -r\Big) \leqslant \frac{\mathbb{E}[M_\tau^+] - \mathbb{E}[M_0]}{r}$$

where $M_t^+ = \max(M_t, 0)$.

Proof. Let $S = \min\{t \in H \colon M_t \geqslant r\}$ with $S = +\infty$ if $M_t < r$ for each $t \in H$, then Theorem 2.29, with $T = \tau$, gives

$$\mathbb{E}[M_\tau] \geqslant \mathbb{E}[M_{S\wedge\tau}] = \mathbb{E}[M_S\mathbb{I}_{\{S<+\infty\}}] + \mathbb{E}[M_\tau\mathbb{I}_{\{S=+\infty\}}]$$

Since $M_\sigma \geqslant r$ we get

$$r\mathbb{P}\Big(\max_{t\in H} M_t \geqslant r\Big) = r\mathbb{P}(S < +\infty) \leqslant \mathbb{E}[M_S\mathbb{I}_{\{S<+\infty\}}] \leqslant \mathbb{E}[M_\tau\mathbb{I}_{\{S<+\infty\}}] \leqslant \mathbb{E}[M_\tau^+\mathbb{I}_{\{S<+\infty\}}] \leqslant \mathbb{E}[M_\tau^+].$$

Let T be $T = \min\{t \in H, M_t \leqslant -r\}$ taking $S = 0$ in Theorem 2.29 we get

$$\mathbb{E}[M_0] \leqslant \mathbb{E}[M_{T\wedge\tau}] = \mathbb{E}[M_T\mathbb{I}_{\{T<+\infty\}}] + \mathbb{E}[M_\tau\mathbb{I}_{\{T=+\infty\}}].$$

Thus we get

$$\begin{aligned}-r\mathbb{P}\Big(\min_{t\in H} M_t \leqslant -r\Big) = -r\mathbb{P}(T < +\infty) \geqslant \mathbb{E}[M_T\mathbb{I}_{\{T<+\infty\}}] &\geqslant \mathbb{E}[M_0] - \mathbb{E}[M_\tau\mathbb{I}_{\{T=+\infty\}}]\\ &\geqslant \mathbb{E}[M_0] - \mathbb{E}[M_\tau^+].\end{aligned}$$

□

Theorem 2.31. *Let M be a cadlag submartingale and fix $0 < \tau < +\infty$. Then for any $r > 0$ we have*

$$\mathbb{P}\Big(\max_{t\in[0,\tau]} M_t \geqslant r\Big) \leqslant \frac{\mathbb{E}[M_\tau^+]}{r}$$

$$\mathbb{P}\Big(\min_{t\in[0,\tau]} M_t \geqslant r\Big) \leqslant \frac{\mathbb{E}[M_\tau^+] - \mathbb{E}[M_0]}{r}.$$

Proof. Let H be a countable dense subset of $[0,\tau]$ that contains 0 and τ, and let $H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots \subset H$ be finite sets such that $\bigcup_{n\in\mathbb{N}} H_n = H$. From Lemma 2.30 we have for any $b < r$, since M_t is cadlag,

$$\begin{aligned}\mathbb{P}\Big(\max_{t\in[0,\tau]} M_t > b\Big) = \mathbb{P}\Big(\max_{t\in H} M_t > b\Big) &= \lim_{n\to+\infty}\mathbb{P}\Big(\max_{t\in H_n} M_t > b\Big)\\ &\leqslant \frac{\mathbb{E}[M_\tau^+]}{b}.\end{aligned}$$

Taking $b \to r$, the first inequality is proved. The second inequality can be proved in a similar way. □

Theorem 2.32. *Let M_t be a cadlag nonnegative submartingale and fix $0<\tau<+\infty$. Then for any $p>1$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,\tau]} M_t^p\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}[M_\tau^p].$$

Proof. Hereafter we write $M_t^* = \sup_{s\in[0,t]} M_s$. The first step of the proof is to establish the following inequality, for any $r>0$, we have

$$\mathbb{P}(M_\tau^*>r) \leqslant \frac{\mathbb{E}[M_\tau \mathbb{I}_{M_\tau^* \geqslant r}]}{r}. \tag{2.2}$$

Let $T_r\colon \Omega \to \mathbb{R}_+$ be the function defined as

$$T_r = \inf\{t>0, M_t>r\}.$$

Since $T_r = \tilde{\tau}_G$ (i.e. T is the hitting time of the set G) of the open set $G=(r,+\infty)\subset \mathbb{R}_+$, then, by Proposition 2.19, T_r is a $\mathcal{F}_{t+}$-stopping time, and so a $\mathcal{H}_t$-stopping time (where $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ is the right-continuous and completed enlargement of $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$). Since M_t is cadlag we have that $M_{T_r}\geqslant r$, furthermore if $M_\tau^*>r$ then $T_r\leqslant\tau$, and thus

$$r\mathbb{P}(M_\tau^*>r) \leqslant \mathbb{E}[M_{T_r}\mathbb{I}_{\{M_\tau^*>r\}}] \leqslant \mathbb{E}[M_{T_r}\mathbb{I}_{\{T_r\leqslant\tau\}}].$$

Since M_t is a cadlag $\mathcal{F}_t$-submartingale, by Theorem 2.11, it is also a cadlag $\mathcal{H}_t$-submartingale, thus Theorem 2.29 gives

$$\begin{aligned}
\mathbb{E}[M_{T_r}\mathbb{I}_{\{T_r\leqslant\tau\}}] &= \mathbb{E}[M_{\tau\wedge T_r}] - \mathbb{E}[M_\tau \mathbb{I}_{\{T_r>\tau\}}]\\
&\leqslant \mathbb{E}[M_\tau] - \mathbb{E}[M_\tau\mathbb{I}_{\{T_r>\tau\}}] = \mathbb{E}[M_\tau \mathbb{I}_{\{T_r\leqslant\tau\}}]\\
&\leqslant \mathbb{E}[M_\tau \mathbb{I}_{\{M_\tau^*\geqslant r\}}],
\end{aligned}$$

and this verifies inequality (2.2).

Consider $0<b<+\infty$ and let $\mu_{M_\tau^*}$ be the probability law of M_τ^*

$$\begin{aligned}
\mathbb{E}[(M_\tau^*\wedge b)^p] &= \int_0^{+\infty} (x\wedge b)^p \mu_{M_\tau^*}(\mathrm{d}x)\\
&= \int_0^b x^p \mu_{M_\tau^*}(\mathrm{d}x) + b^p \mathbb{P}(M_\tau^*>b)\\
&= \left(-x^p \int_x^{+\infty} \mu_{M_\tau^*}(\mathrm{d}y)\right)_{x=0}^{x=b} + p\int_0^b x^{p-1}\left(\int_x^{+\infty}\mu_{M_\tau^*}(\mathrm{d}y)\right)\mathrm{d}x + b^p\mathbb{P}(M_\tau^*>b)\\
&= -b^p\mathbb{P}(M_\tau^*>b) + p\int_0^b x^{p-1}\mathbb{P}(M_\tau^*>x)\mathrm{d}x + b^p\mathbb{P}(M_\tau^*>b)\\
&= p\int_0^b x^{p-1}\mathbb{P}(M_\tau^*>x)\mathrm{d}x.
\end{aligned}$$

So, by Holder inequality, we get

$$\begin{aligned}
\mathbb{E}[(M_\tau^*\wedge b)^p] &= \int_0^b p x^{p-1}\mathbb{P}(M_\tau^*>x)\mathrm{d}x\\
&\leqslant \int_0^b p x^{p-2}\mathbb{E}[M_\tau \mathbb{I}_{\{M_\tau^*\geqslant x\}}]\mathrm{d}x\\
&= \mathbb{E}\left[M_\tau \int_0^{b\wedge M_\tau^*} p x^{p-2}\mathrm{d}x\right] = \frac{p}{p-1}\mathbb{E}[M_\tau (M_\tau^*\wedge b)^{p-1}]\\
&\leqslant \frac{p}{p-1}\mathbb{E}[M_\tau^p]^{\frac{1}{p}}\,\mathbb{E}[(M_\tau^*\wedge b)^p]^{\frac{p-1}{p}}.
\end{aligned}$$

Since $0<\mathbb{E}[(M_\tau^*\wedge b)^p]^{\frac{p-1}{p}}\leqslant b$, we can divide the previous inequality by $\mathbb{E}[(M_\tau^*\wedge b)^p]^{\frac{p-1}{p}}$ obtaining that

$$\mathbb{E}[(M_\tau^*\wedge b)^p]^{\frac{1}{p}} \leqslant \left(\frac{p}{p-1}\right)\mathbb{E}[M_\tau^p]^{\frac{1}{p}}.$$

Rising both sides to the power p and taking $b \to +\infty$, by Monotone convergence theorem, we get the thesis. □

We want to provide an important application of the Theorem 2.32.

Theorem 2.33. (Fernique's theorem for Brownian motion) *Fix $\tau \in \mathbb{R}_+$, then there for any $C_\tau < \frac{1}{2\tau}$ we have*

$$\mathbb{E}\left[\exp\left(C_\tau\left(\sup_{t\in[0,\tau]}|B_t|\right)^2\right)\right] = \mathbb{E}[\exp(C_\tau B_\tau^{*2})] < +\infty.$$

Proof. We have that

$$\begin{aligned}\mathbb{E}\left[\exp\left(C_\tau\left(\sup_{t\in[0,\tau]}|B_t|\right)^2\right)\right] &= \sum_{n=0}^{+\infty}\frac{C_\tau^n}{n!}\mathbb{E}\left[\left(\sup_{t\in[0,\tau]}|B_t|\right)^{2n}\right]\\ &\leqslant \sum_{n=0}^{+\infty}\frac{C_\tau^n}{n!}\left(\frac{2n}{2n-1}\right)^{2n}\mathbb{E}[|B_\tau|^{2n}].\end{aligned}$$

We have that

$$\left(\frac{2n}{2n-1}\right)^n = \left(1+\frac{1}{2n-1}\right)^n \to e, \quad n\to+\infty,$$

which implies that there is $K>0$ for which

$$\sup_{n\in\mathbb{N}}\left(\frac{2n}{2n-1}\right)^n \leqslant K.$$

Furthermore we have that

$$\mathbb{E}[|B_\tau|^{2n}] = \frac{1}{\sqrt{(2\pi)\tau}}\int_{\mathbb{R}} e^{-\frac{x^2}{2\tau}}x^{2n}\mathrm{d}x = \frac{2}{\sqrt{(2\pi)\tau}}\int_0^{+\infty} e^{-y}y^{\frac{2n-1}{2}}\mathrm{d}y = \frac{2^{n-\frac{1}{2}}\tau^n}{\sqrt{2\pi}}\Gamma\left(\frac{2n+1}{2}\right) =$$

$$= 2^n\frac{(2n)!}{2^{2n}n!} = \frac{(2n)!\tau^n}{2^n n!}.$$

This implies that

$$\mathbb{E}\left[\exp\left(C_\tau\left(\sup_{t\in[0,\tau]}|B_t|\right)^2\right)\right] \leqslant K\sum_{n=0}^{+\infty}\frac{C_\tau^n(2n)!}{2^n(n!)^2}\tau^n.$$

Using Stirling approximation we have that

$$\frac{(2n)!}{(n!)^2} \sim \frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{(n)^{2n}e^{-2n}2\pi n} \sim \frac{2^{2n}}{\pi\sqrt{n}},$$

as $n\to+\infty$, which implies that there exists a constant $K'>0$ for which

$$\frac{(2n)!}{(n!)^2} \leqslant K' 2^{2n}.$$

Thus we obtain

$$\mathbb{E}\left[\exp\left(C_\tau\left(\sup_{t\in[0,\tau]}|B_t|\right)^2\right)\right] \leqslant K\sum_{n=0}^{+\infty} 2^n C_\tau^n \tau^n$$

which is convergent whenever $C_\tau \leqslant \frac{1}{2\tau}$. □

Chapter 3
Continuous (local) martingales

3.1 The space of continuous L^2 martingales

In this section we want to consider continuous $L^2(\Omega)$ martingales.

Definition 3.1. *Let M_t be a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, we say that M_t is a continuous martingales (bounded) in L^2 if for any $\omega\in\Omega$ the map $t\mapsto M_t(\omega)$ is continuous and if for any $t\in\mathbb{R}_+$, $\mathbb{E}[M_t^2]<+\infty$. We denote by $\mathcal{M}_c^2(\{\mathcal{F}_t\}_{t\in\mathbb{R}_+})$ (or simply $\mathcal{M}_c^2$) the set of L^2 continuous martingales.*

On $\mathcal{M}_c^2$ we define the function

$$d_{\mathcal{M}_c^2}(M,N)=\sum_{n\in\mathbb{N}}2^{-n}(1\wedge(\mathbb{E}[|M_n-N_n|^2])^{1/2}).$$

The function $\mathcal{M}_c^2$ is not a distance on $\mathcal{M}_c^2$. Indeed, although it is positive, symmetric and satisfies triangular inequality it is not true that $d_{\mathcal{M}_c^2}(M,N)=0$ if and only if $N=M$. In any case a positive, weak, result holds.

Proposition 3.2. *Consider $M,N\in\mathcal{M}_c^2$ then $d_{\mathcal{M}_c^2}(M,N)=0$ if and only if M and N are indistinguishable.*

Proof. If $d_{\mathcal{M}_c^2}(M,N)=0$ then for any $n\in\mathbb{N}$ we have $\mathbb{E}[(M_n-N_n)^2]=0$. Since M,N are martingales then $(M_t-N_t)^2$ is a submartingale (since is the composition of the martingale $M-N$ with a convex function), which implies that

$$\mathbb{E}[(M_t-N_t)^2]\leqslant\mathbb{E}[(M_{\lfloor t\rfloor+1}-N_{\lfloor t\rfloor+1})^2]=0.$$

This implies that for any $t\in\mathbb{R}_+$ then $\mathbb{P}(M_t=N_t)=1$, and so N is a modification of M. Since both N and M are continuous, and thus they are cadlag and for cadlag martingales the a process is a modification of another if and only if they are indistinguishable then M and N are indistinguishable. □

Proposition 3.2 suggests to consider the space

$$\hat{\mathcal{M}}_c^2=\mathcal{M}_c^2/\sim_{\text{ind}}$$

where $\sim_{\text{ind}}$ is the equivalence relation for which $N\sim_{\text{ind}}M$ if the process N is indistinguishable from M. By Proposition 3.2 the function $d_{\mathcal{M}_c^2}$ is compatible with respect the the equivalence relation $\sim_{\text{ind}}$ and so it pass to the quotient. With an abuse of notation we denote again by $d_{\mathcal{M}_c^2}$ the function $d_{\mathcal{M}_c^2}$ on the set $\hat{\mathcal{M}}_c^2$.

Remark 3.3. It is important to note that by Theorem 2.32 an equivalent metric on $\hat{\mathcal{M}}_c^2$ is given by

$$\tilde{d}_{\mathcal{M}_c^2}(M,N)=\sum_{n\in\mathbb{N}}2^{-n}\left(\mathbb{E}\left[\sup_{t\in[n-1,n]}|M_t-N_t|^2\right]^{1/2}\wedge 1\right).$$

Theorem 3.4. *The set $\hat{\mathcal{M}}_c^2(\{\mathcal{G}_t\}_{t\in\mathbb{R}_+})$ of L^2 continuous martingales with respect to the continuous filtration $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ is a complete metric space with respect to $d_{\mathcal{M}_c^2}$.*

Proof. The function $d_{\mathcal{M}_c^2}$ is positive, symmetric and satisfies the triangular inequality and for Proposition 3.2 $d_{\mathcal{M}_c^2}(\{N\},\{M\})$ if and only if $M\sim_{\text{ind}}N$ (and so they are the same object in $\hat{\mathcal{M}}_c^2$).

What remains to prove is that $\hat{\mathcal{M}}_c^2$ is complete with respect to $d_{\mathcal{M}_c^2}$. Let $\{M^n\}$ be a Cauchy sequence in $\hat{\mathcal{M}}_c^2$ (where M^n is some sequence in $\mathcal{M}_c^2$ of on of their representative), then we have that for any for any $t\in\mathbb{R}_+$

$$(1\wedge\mathbb{E}[|M_t^k-M_t^h|^2])^{1/2}\leqslant(1\wedge\mathbb{E}[|M_{\lfloor t\rfloor+1}^k-M_{\lfloor t\rfloor+1}^h|^2])^{1/2}\leqslant 2^{\lfloor t\rfloor+1}d_{\mathcal{M}_c^2}(M^k,M^h).$$

It follows that for each $t\in\mathbb{R}_+$, the sequence $\{M_t^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$. This means that for each $t\in\mathbb{R}_+$ there is an adapted process Y_t such that $M_t^k\to Y_t$ in $L^2(\Omega)$. Furthermore if $\tilde{Y}_t$ is any adapted modification of Y_t (which is equivalent to any modification since the set of measure 0 are inside $\mathcal{G}_t$ being $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ a complete filtration) we have also $M_t^k\to\tilde{Y}_t$ in $L^2(\Omega)$. Furthermore the process Y_t (and so any modification of it) are $\mathcal{G}_t$ martingale. Indeed for any $A\in\mathcal{G}_s$ we have that

$$\mathbb{E}[\mathbb{I}_AY_t]=\lim_{k\to+\infty}\mathbb{E}[\mathbb{I}_AM_t^k]=\lim_{k\to+\infty}\mathbb{E}[\mathbb{I}_AM_s^k]=\mathbb{E}[\mathbb{I}_AY_s]$$

from which we can conclude that Y_t is a martingale.

In general $Y_t\notin\mathcal{M}_c^2$ since it can happen that $t\mapsto Y_t(\omega)$ is not continuous. We want now to prove that there is a modification of Y_t which is continuous. By Remark 3.3 we can choose M^{n_k} a subsequence of M_t^n for which

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant k}|M_t^{n_k}-M_t^{n_{k+1}}|>2^{-k}\right)\leqslant 2^{-k}.$$

Indeed consider an increasing sequence $n_k\uparrow+\infty$ such that

$$\tilde{d}_{\mathcal{M}_c^2}(M^n,M^m)\leqslant 2^{-3k}$$

for any $n,m\geqslant n_k$. Then we have that

$$\left(1\wedge\mathbb{E}\left[\sup_{1\leqslant t\leqslant k}|M_t^n-M_t^m|^2\right]\right)^{1/2}\leqslant 2^k\tilde{d}_{\mathcal{M}_c^2}(M^n,M^m)\leqslant 2^{-2k}$$

for $n,m\geqslant n_k$. Thus, by Markov inequality we have

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant k}|M_t^{n_k}-M_t^{n_{k+1}}|>2^{-k}\right)\leqslant 2^{2k}\mathbb{E}\left[\sup_{1\leqslant t\leqslant k}|M_t^n-M_t^m|^2\right]\leqslant 2^{2k-4k}\leqslant 2^{-2k}.$$

Since

$$\sum_{k=1}^{+\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant k}|M_t^{n_k}-M_t^{n_{k+1}}|>2^{-k}\right)\leqslant\sum_{k=1}^{+\infty}2^{-k}<+\infty$$

for Borel-Cantelli lemma

$$\Omega_1=\left(\limsup_{k\to+\infty}\left\{\sup_{0\leqslant t\leqslant k}|M_t^{n_k}-M_t^{n_{k+1}}|>2^{-k}\right\}\right)^c=\liminf_{k\to+\infty}\left\{\sup_{0\leqslant t\leqslant k}|M_t^{n_k}-M_t^{n_{k+1}}|\leqslant 2^{-k}\right\}$$

has measure 1. This means that for any for any $\tau>0$ and $\omega\in\Omega_1$ the sequence of function $t\mapsto M_t^{n_k}(\omega)$ is a Cauchy sequence in $C^0([0,\tau],\mathbb{R})$, and so for any $\omega\in\Omega_1$ there is a unique continuous function $t\mapsto M_t(\omega)$ such that $M_t^{n_k}(\omega)\to M_t(\omega)$ uniformly on compact subsets of $\mathbb{R}_+$. Consider the process

$$M_t(\omega)=\begin{cases}\lim_{k\to+\infty}M_t^{n_k}(\omega) & \omega\in\Omega_1\\ 0 & \omega\notin\Omega_1\end{cases}$$

then $M_t(\omega)$ is a continuous process, it is adapted (since $\Omega_1\in\mathcal{G}_0\subset\mathcal{G}_t$ being a set of full measure) and $M_t^{n_k}\to M_t$ almost surely. This implies that M_t is a modification of Y_t, thus $M_t\in\mathcal{M}_c^2$ and

$$\mathbb{E}[\lfloor M_k-M_k^n|^2]\to 0$$

as $n\to+\infty$ and for any $k\in\mathbb{N}$. This implies that $d_{\mathcal{M}_c^2}(M,M^n)\to 0$ as $n\to+\infty$ which proves the thesis. □

Remark 3.5. It is possible to generalize the previous theorem considering $\mathcal{M}_{\text{cadlag}}^2$, i.e. the set of $L^2(\Omega)$ cadlag martingales, instead of $\mathcal{M}_c^2$. Theorem 3.4 implies that $\mathcal{M}_c^2$ is a closed subspace of $\mathcal{M}_{\text{cadlag}}^2$.

If X is a (progressive) process and T is a stopping time, then we denote by

$$X_t^T = X_{t\wedge T}$$

the process X stopped at the stopping time T.

Definition 3.6. *Let M be an adapted continuous process. We say that M is a local martingale if there is a sequence of stopping time $T_1 \leqslant T_2 < \cdots \leqslant T_n \leqslant \cdots$ such that $T_n \to +\infty$ almost surely as $n \to +\infty$ and $M_t^{T_n}$ is a continuous martingale. Hereafter, we call a sequence of stopping times $\{T_n\}_{n\in\mathbb{N}}$ with the previous properties* localization stopping times.

Remark 3.7. If M_0 is $L^2(\Omega)$ and since we only consider *continuous* local martingale, we can always consider the localization sequence $\{T_n\}_{n\in\mathbb{N}}$ such that M^{T_n} is a $\mathcal{M}_c^2$ martingale. Indeed, consider

$$S_n = \inf\{t \geqslant 0, |M_t| > n\},$$

then

$$|M_t^{S_n\wedge T_n}| \leqslant |M_0| \vee n \in L^2(\Omega).$$

In the same way if $M_0 \in L^\infty(\Omega)$ we can suppose that M^{T_n} is a bounded martingale.

3.2 Bounded variation processes

Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. For any $t \in \mathbb{R}_+$ we denote by $\Pi([0,t])$ the set of (finite) partitions of the interval $[0,t]$. It is important to note that $\Pi([0,t])$ has a partial order given by the inclusion $\pi \subset \pi'$. Furthermore, for any two partition π, π', the union $\pi \cup \pi'$ is the smallest partition containing both π, π' and the intersection $\pi \cap \pi'$ is the biggest partition which is contained in both π and π'. We can defined also the diameter of a partition as

$$|\pi| = \max\{|t_i - t_{i-1}|, t_i \in \pi \setminus \{0\}\}.$$

Finally, we denote by $\Pi([0,\infty))$ the set of partitions of $[0,+\infty)$ which are locally finite, i.e. if $\pi \in \Pi([0,\infty))$ then $\pi^t = (\pi \cap [0,t]) \cup \{t\}$ is a finite partition of $[0,t]$.

Definition 3.8. *Let $F: \Pi([0,t]) \to \mathbb{R}$ be a function we say that the limit*

$$\lim_{|\pi|\to 0} F(\pi)$$

exists if, for any sequence $\pi_1 \subset \pi_2 \subset \cdots \subset \pi_n \subset \cdots \in \Pi([0,t])$ of increasing partitions of $[0,t]$ such that $|\pi_n| \to 0$, as $n \to +\infty$, the limit $\lim_{n\to+\infty} F(\pi_n)$ exists and it does not depend on the sequence. In this case, we write

$$\lim_{|\pi|\to 0} F(\pi) = \lim_{n\to+\infty} F(\pi_n).$$

Let f be a measurable function, we denote by

$$V_t(f) := \sup_{\pi\in\Pi([0,t])} \left(\sum_{t_i\in\pi\setminus\{0\}} |f(t_i) - f(t_{i-1})| \right) =: \sup_{\pi\in\Pi([0,t])} V_t^\pi(f)$$

the variation of the function f in the interval $[0,t]$.

Remark 3.9. By triangular inequality, if $\pi \subset \pi' \in \Pi([0,t])$ then

$$V_t^\pi(f) \leqslant V_t^{\pi'}(f).$$

Definition 3.10. *A function f is said to have* bounded variation *on $\mathbb{R}_+$ if, for any $t \in \mathbb{R}_+$, we have $V_t(f) < +\infty$.*

Definition 3.11. *Consider two Borel functions $f, g: \mathbb{R}_+ \to \mathbb{R}$ we define the* Riemann-Stieltjes integral $\int_0^t g(t)\mathrm{d}f(t)$, *as the following limit*

$$\int_0^t g(t)\mathrm{d}f(t) = \lim_{|\pi|\to 0} \sum_{t_i\in\pi\setminus\{0\}} g(\xi_i)(f(t_i) - f(t_{i-1})),$$

where ξ_i is any point $t_{i-1} \leqslant \xi_i < t_i$.

Herafter if $\pi \in \Pi([0,t])$ and $\bar{\xi} = \{\xi_{t_i}\}_{t_i \in \pi \setminus \{0\}}$ such that $t_{i-1} \leqslant \xi_{t_i} < t_i$, g, f are two Borel function we define

$$\Gamma(\pi, \bar{\xi}, g, f) = \sum_{t_i \in \pi \setminus \{0\}} g(\xi_{t_i})(f(t_i) - f(t_{i-1})).$$

Theorem 3.12. *Let g be a continuous function and let f be a right-continuous function with bounded variation, then the Riemann-Stieltjes integral is well defined.*

Proof. Let $\pi \subset \pi'$ be two partitions, then there is a map

$$I^{\pi',\pi}: \pi' \to \pi$$

associating with any $t'_{j'} \in \pi'$ the $t_i \in \pi$ which is the biggest $t_i \in \pi$ that is less or equal then $t_i \leqslant t'_{j'}$. Using this notation, we have

$$|\Gamma(\pi, \bar{\xi}, g, f) - \Gamma(\pi', \bar{\xi}', g, f)| = \left| \sum_{t_i \in \pi \setminus \{0\}} g(\xi_{t_i})(f(t_i) - f(t_{i-1})) - \sum_{t'_{j'} \in \pi' \setminus \{0\}} g(\xi'_{t'_j})(f(t'_j) - f(t'_{j-1})) \right| =$$

$$= \left| \sum_{t'_{j'} \in \pi' \setminus \{0\}} \left(g\left(\xi_{I^{\pi',\pi}(t'_{j'})} \right) - g(\xi_{t'_j}) \right) (f(t'_{j'}) - f(t'_{j'-1})) \right| \leqslant$$

$$\leqslant \left(\max_{t'_{j'} \in \pi' \setminus \{0\}} \left| g\left(\xi_{I^{\pi',\pi}(t'_{j'})} \right) - g(\xi_{t'_{j'}}) \right| \right) \left(\sum_{t'_{j'} \in \pi'} (f(t'_{j'}) - f(t'_{j'-1})) \right)$$

$$\leqslant \left(\max_{t_i \in \pi \setminus \{0\}} \left(\sup_{t,s \in [t_{i-1}, t_i]} |g(t) - g(s)| \right) \right) V_t(f).$$

If $\pi_1 \subset \cdots \subset \cdots \subset \pi_n \subset \cdots \in \Pi([0,t])$ is any increasing sequence such that $|\pi_n| \to 0$, since

$$\max_{t_i \in \pi_n \setminus \{0\}} \left(\sup_{t,s \in [t_{i-1}, t_i]} |g(t) - g(s)| \right) \to 0, \quad n \to +\infty,$$

being g uniformly continuous on $[0,t]$, the sequence $\{\Gamma(\pi_n, \bar{\xi}, g, f)\}_{n \in \mathbb{N}}$ is a Cauchy sequence on $\mathbb{R}$ which has a unique limit. If now $\{\pi'_n\}_{n \in \mathbb{N}}$ is another increasing sequence by the previous computation we have

$$|\Gamma(\pi_n, \bar{\xi}, g, f) - \Gamma(\pi'_n, \bar{\xi}', g, f)| \leqslant$$

$$\leqslant \left| \sum_{t_i \in \pi_n \setminus \{0\}} g(\xi_{t_{i-1}})(f(t_i) - f(t_{i-1})) - \sum_{t'_j \in \pi'_n \cup \pi_n \setminus \{0\}} g(\xi'_{t'_j})(f(t'_j) - f(t'_{j-1})) \right| +$$

$$+ \left| \sum_{t_i \in \pi'_n \setminus \{0\}} g(\xi_{t_{i-1}})(f(t_i) - f(t_{i-1})) - \sum_{t'_j \in \pi'_n \cup \pi_n \setminus \{0\}} g(\xi'_{t'_j})(f(t'_j) - f(t'_{j-1})) \right| \leqslant$$

$$\leqslant \left(\max_{t_i \in \pi_n \setminus \{0\}} \left(\sup_{t,s \in [t_{i-1}, t_i]} |g(t) - g(s)| \right) + \max_{t_i \in \pi'_n \setminus \{0\}} \left(\sup_{t,s \in [t_{i-1}, t_i]} |g(t) - g(s)| \right) \right) V_t(f).$$

This means that $\{\Gamma(\pi_n, \bar{\xi}, g, f)\}_{n \in \mathbb{N}}$ and $\{\Gamma(\pi'_n, \bar{\xi}', g, f)\}_{n \in \mathbb{N}}$ have the same limit which is the Riemmann-Stieltjes integral $\int_0^t g(t) \mathrm{d} f(t)$. □

For the following it is useful the following lemma.

Proposition 3.13. *Suppose that f is a cadlag bounded variation function, then the map $t \mapsto V_t(f)$ is a cadlag function and*

$$V_t(f) - V_{t-}(f) = |f(t) - f(t-)|.$$

Proof. We give here only a sketch of the proof for a detailed treatment of the topic see Chapter 2 and Chapter 10 of [8].

For any $a,b\in\mathbb{R}_+$ we denote by $\Pi([a,b])$ the partitions of $[a,b]$ and we write

$$V(f,[a,b])=\sup_{\pi\in\Pi([a,b])}\sum_{t_i\in\pi\setminus\{a\}}|f(t_i)-f(t_{i-1})|.$$

It is possible to prove that for any $c\in(a,b)$ we have

$$V(f,[a,b])=V(f,[a,c])+V(f,[c,b])$$

and obviously $V_t(f)=V(f,[0,t])$, thus

$$V_t(f)=V_{t_0}(f)-V(f,[t,t_0])$$

and so $\lim_{s\to t_+}V_t(f)=V_{t_0}(f)-\lim_{s\to t_+}V(f,[s,t_0])$, and so $V_t(f)$ is right continuous if $\lim_{s\to t+}V(f,[s,t_0])=V(f,[t,t_0])$. If $\pi\in\Pi([t,t_0])$ is any partition of $[t,t_0]$ let $t_{\pi,\min}$ the first element in π after t then

$$V^{\pi}(f,[t,t_0])-|f(t)-f(t_{\min})|=V^{\pi\setminus\{t\}}(f,[t_{\min},t_0])\leqslant V(f,[t_{\min},t_0])$$

From which we get

$$V^{\pi}(f,[t,t_0])-V(f,[t_{\min},t_0])\leqslant|f(t)-f(t_{\min})|.$$

Taking the limit $|\pi|\to 0$ we get

$$0\leqslant V(f,[t,t_0])-\lim_{s\to t+}V(f,[s,t_0])\leqslant\lim_{s\to t_+}|f(t)-f(s)|=0$$

since f is cadlag. In a similar way if $\pi\in\Pi([0,t])$ and being $t_{\pi,\max}$ the first element in π before t we have

$$V_t^{\pi}(f)-|f(t)-f(t_{\pi,\max})|\leqslant V_{t_{\pi,\max}}(f)$$

to which we get $V_t(f)-V_{t-}(f)\leqslant|f(t)-f(t-)|$. In a similar way

$$V_{t_{\pi,\max}}^{\pi\setminus\{t\}}(f)+|f(t)-f(t_{\pi,\max})|\leqslant V_t(f)$$

taking $\pi\setminus\{t\}=\tilde{\pi}\in\Pi([0,t_{\pi,\max}])$ we get

$$|f(t)-f(t_{\pi,\max})|\leqslant V_t(f)-V_{t_{\pi,\max}}^{\pi\setminus\{t\}}(f)\to V_t(f)-V_{t_{\pi,\max}}(f)$$

and so taking $t_{\pi,\max}\to t$ we get $|f(t)-f(t-)|\leqslant V_t(f)-V_{t-}(f)$. □

Remark 3.14. An important consequence of Proposition 3.13 is that if f is a *continuous* function with bounded variation then the function $t\mapsto V_t(f)$ is an *increasing continuous function.*

Definition 3.15. *A continuous adapted process X on $\mathbb{R}_+$ has* bounded variation *if $V_t(X_\cdot(\omega))<+\infty$ almost surely.*

Theorem 3.16. *Let M_t be a continuous local martingale with bounded variation, then $M_t=M_0$ almost surely.*

Proof. Without loss of generality, we can suppose that $M_t\in\mathcal{M}_c^2$ and that $V_s(M)<C$ for some $C>0$. Indeed, if T_n is the localization sequence making for which M^{T_n} is a martingale and we denote by

$$S_n=\inf\{t\geqslant 0,|M_t|\geqslant n\}\wedge\inf\{t\geqslant 0,V_t(M_\cdot)\geqslant n\},$$

then also $T_n\wedge S_n$ is a localization sequence and $M^{T_n\wedge S_n}\mathbb{I}_{\{M_0\leqslant n\}}$ is a $\mathcal{M}_c^2$ (more precisely, bounded) martingale and $V_t(M_\cdot)\leqslant n$ (we recall that by Proposition 3.13 and Remark 3.14 the function $V_t(M_\cdot)$ is continuous and so $\inf\{t\geqslant 0,V_t(M_\cdot)\geqslant n\}$ is a $\mathcal{F}_t$ stopping time). If the theorem holds for $M^{T_n\wedge S_n}\mathbb{I}_{\{|M_0|\leqslant n\}}$, i.e. $M_t^{T_n\wedge S_n}=M_0$ on $|M_0|\leqslant n$ almost surely, then

$$M_t=\lim_{n\to+\infty}M_t^{T_n\wedge S_n}=M_0$$

almost surely. Thus, it is enough to prove the theorem for M with the previous conditions.

Since M_t is a martingale, for any $\pi\in\Pi([0,t])$, we have that

$$\mathbb{E}[(M_t-M_0)^2]=\mathbb{E}[M_t^2]-\mathbb{E}[M_0^2]=\mathbb{E}\left[\sum_{t_i\in\pi\setminus\{0\}}M_{t_i}^2-M_{t_{i-1}}^2\right]=\mathbb{E}\left[\sum_{t_i\in\pi\setminus\{0\}}(M_{t_i}-M_{t_{i-1}})^2\right],$$

where we used the fact that $\mathbb{E}[M_{t_i}M_{t_{i-1}}] = \mathbb{E}[\mathbb{E}[M_{t_i}|\mathcal{F}_{t_i}]M_{t_{i-1}}] = \mathbb{E}[M_{t_{i-1}}^2]$. Thus we have that

$$\begin{aligned}\mathbb{E}[(M_t - M_0)^2] &= \mathbb{E}\left[\sum_{t_i \in \pi\setminus\{0\}} (M_{t_i} - M_{t_{i-1}})^2\right] \\ &\leqslant \mathbb{E}\left[\sup_{t_i\in\pi\setminus\{0\}} |M_{t_i} - M_{t_{i-1}}|\left(\sum_{t_i\in\pi\setminus\{0\}} |M_{t_i} - M_{t_{i-1}}|\right)\right] \\ &\leqslant \mathbb{E}\left[\sup_{t_i\in\pi\setminus\{0\}} |M_{t_i} - M_{t_{i-1}}| V_t(M.)\right] \leqslant C\mathbb{E}\left[\sup_{t_i\in\pi\setminus\{0\}} |M_{t_i} - M_{t_{i-1}}|\right].\end{aligned}$$

By Doob's martingale inequality we have that $\sup_{t_i\in\pi\setminus\{0\}} |M_{t_i} - M_{t_{i-1}}| \leqslant \sup_{s\in[0,t]} |M_s|$ which is an $L^2(\Omega)$ (and thus $L^1(\Omega)$) random variable. Furthermore, since M_s is continuous, and thus uniformly continuous on $[0,t]$, $\lim_{|\pi|\to 0} \sup_{t_i\in\pi\setminus\{0\}} |M_{t_i} - M_{t_{i-1}}| = 0$. By Lebesgue dominated convergence theorem, this implies that

$$\mathbb{E}[(M_t - M_0)^2] \leqslant \lim_{|\pi|\to 0} C\mathbb{E}\left[\sup_{t_i\in\pi\setminus\{0\}} |M_{t_i} - M_{t_{i-1}}|\right] = 0. \qquad \square$$

3.3 Quadratic variation of local martingales

Definition 3.17. *Let M_t be a continuous local martingale we say that the continuous adapted increasing process $[M]_t$ is the quadratic variation of M_t if $[M]_0 = 0$ and $M_t^2 - [M]_t$ is a local martingale.*

We first we establish that if quadratic variation of a local martingale is unique.

Proposition 3.18. *Suppose that M_t is a local martingale then if the quadratic variation $[M]$ exists is unique up to indistiguishability.*

Proof. Let K and K' two processes which are continuous, adapted and increasing and such that $M^2 - K$ and $M^2 - K'$ are local martingales. Then we have that

$$K' - K = M^2 - K - M^2 + K'$$

is a local martingale, being the sum of local martingales. Furthermore, since increasing continuous processes have bounded variation, $K' - K$ has also bounded variation. This means that $K' - K$ is a continuous local martingale with bounded variation, which implies that $K_t - K'_t = K_0 - K'_0 = 0$ almost surely. This means that K is indistinguishable from K'. $\square$

Theorem 3.19. *Let M be a continuous local martingale, then there is one (up to indistiguishability) continuous increasing process $[M,M]$ such that $\{M_t^2 - [M,M]_t\}_{t\in\mathbb{R}_+}$ is a local martingale. Furthermore, if $\pi^1 \subset \cdots \subset \pi^{n+1} \subset \cdots$ we have that*

$$[M,M]_t = \lim_{n\to+\infty} \sum_{t_i\in\pi^n} (M_{t_i\wedge t} - M_{t_{i-1}\wedge t})^2 \tag{3.1}$$

in probability.

3.3.1 A special version of the theorem

We will prove first a special version of the previous theorem (with a stronger kind of convergence).

Proposition 3.20. *Let M be a continuous martingale such that $\mathbb{E}[|M_t|^4] < +\infty$ for any $t \in \mathbb{R}_+$, then there is a unique increasing continuous process $[M,M]_t$ such that $M_t - [M,M]_t \in \mathcal{M}_c^2$ and for any increasing if $\pi_1 \subset \cdots \subset \pi_{n+1} \subset \cdots$ we have that*

$$[M,M]_t = \lim_{n\to+\infty} \sum_{t_i\in\pi^n\setminus\{0\}} (M_{t_i\wedge t} - M_{t_{i-1}\wedge t})^2$$

in $L^2(\Omega)$.

Lemma 3.21. *Let M_t be a martingale and consider $t_1 < t_2$ and $t_3 < t_4$ then $\mathbb{E}[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t_1})]$ is nonzero if and only if $[t_1,t_2] \cap [t_3,t_4] \neq \emptyset$, in that case we have*

$$\mathbb{E}[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t_3})|\mathcal{F}_{t_{\min}}] = \mathbb{E}[M_{t_{\mathrm{fin}}}^2 - M_{t_{\mathrm{in}}}^2|\mathcal{F}_{t_{\min}}],$$

where $[t_1, t_2] \cap [t_3, t_4] = [t_{\text{fin}}, t_{\text{in}}]$ *and* $t_{\min} = \min(t_1, t_2, t_3, t_4)$.

Proof. Suppose that $[t_1, t_2] \cap [t_3, t_4] = \emptyset$ and we can suppose, without loss of generality, that $t_2 < t_3$. Then

$$\begin{aligned}\mathbb{E}[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t_3})] &= \mathbb{E}[(M_{t_2} - M_{t_1})\mathbb{E}[(M_{t_4} - M_{t_3})|\mathcal{F}_{t_2}]] = \\ &= \mathbb{E}[(M_{t_2} - M_{t_1})(M_{t_2} - M_{t_2})] = 0.\end{aligned}$$

Suppose then that $[t_1, t_2] \cap [t_3, t_4] \neq \emptyset$ and we can assume without loss of generality that $t_1 \leqslant \min(t_1, t_2, t_3, t_4)$. Then we have two possibilities: either $[t_1, t_2] \cap [t_3, t_4] = [t_3, t_2]$ or $[t_1, t_2] \cap [t_3, t_4] = [t_3, t_4]$ (i.e. either $t_2 \leqslant t_4$ or $t_4 < t_2$). In the first case we get

$$\begin{aligned}\mathbb{E}[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t_3})|\mathcal{F}_{t_1}] &= \mathbb{E}[M_{t_2}M_{t_4}|\mathcal{F}_{t_1}] - \mathbb{E}[M_{t_2}M_{t_3}|\mathcal{F}_{t_1}] - \mathbb{E}[M_{t_1}M_{t_4}|\mathcal{F}_{t_1}] + \mathbb{E}[M_{t_1}M_{t_3}|\mathcal{F}_{t_1}] \\ &= \mathbb{E}[M_{t_{\text{fin}}}^2|\mathcal{F}_{t_{\min}}] - \mathbb{E}[M_{t_{\text{in}}}^2|\mathcal{F}_{t_{\min}}] - \mathbb{E}[M_{t_1}^2|\mathcal{F}_{t_1}] + \mathbb{E}[M_{t_1}^2|\mathcal{F}_{t_1}]\end{aligned}$$

The other case can be treated in a similar way. □

Let us fix a sequence of partitions $\{0\} \subset \pi^1 \subset \cdots \subset \pi^n \subset \cdots \mathbb{R}_+$ which have a finite number of points when intersected with any bounded subset of $\mathbb{R}_+$. We write

$$\pi^{k,t} = \{\pi^k \cap [0, t]\} \cup \{t\}.$$

If M_t is a process, then we denote by Q_t^{M,π^n} and K_t^{M,π^n} the following continuous processes

$$\begin{aligned}Q_t^{M,\pi^n} &= \sum_{t_i \in \pi^{n,t} \setminus \{0\}} (M_{t_i} - M_{t_{i-1}})^2, \\ K_t^{M,\pi^n} &= \sum_{t_i \in \pi^{n,t} \setminus \{0\}} M_{t_{i-1}}(M_{t_i} - M_{t_{i-1}}).\end{aligned}$$

Remark 3.22. It is important to note that

$$M_t^2 - M_0^2 - 2K_t^{M,\pi^n} = Q_t^{M,\pi^n}.$$

Lemma 3.23. *Suppose that M is a martingale such that $M_t \in L^4(\Omega)$, then there is a constant $C > 0$ (not depending on of π^n) such that*

$$\mathbb{E}[(Q_t^{M,\pi^n})^2] \leqslant \left(\frac{3}{4}\right)^2 C \left(\mathbb{E}\left[\sup_{s \in [0,t]} |M_s|^4\right]\right)^{\frac{1}{2}} \leqslant C\, (\mathbb{E}[M_t^4])^{\frac{1}{2}}.$$

Proof. We have that

$$\begin{aligned}\mathbb{E}[(Q_t^{M,\pi^n})^2] &= \mathbb{E}\left[\sum_{t_j \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}})^4\right] + \\ &\quad + 2\mathbb{E}\left[\sum_{t_j \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}})^2 \left(\sum_{t_j < t_j' \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}'})^2\right)\right] \\ &\leqslant \mathbb{E}\left[\sup_{t_j \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}})^2 \sum_{t_j \in \pi^n} (M_{t_j} - M_{t_{j-1}})^2\right] + \\ &\quad + 2\mathbb{E}\left[\sum_{t_j \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}})^2 \mathbb{E}\left[\sum_{t_j < t_j' \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}'})^2 \middle| \mathcal{F}_{t_j}\right]\right] = \\ &\leqslant \mathbb{E}\left[\sup_{t_j \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}})^4\right]^{\frac{1}{2}} (\mathbb{E}[(Q_t^{M,\pi^n})^2])^{\frac{1}{2}} + \\ &\quad + 2\mathbb{E}\left[\sum_{t_j \in \pi^{n,t}} (M_{t_j} - M_{t_{j-1}})^2 \mathbb{E}[M_t^2 - M_{t_j}^2|\mathcal{F}_{t_j}]\right] \\ &\leqslant \left(\mathbb{E}\left[\sup_{s \in [0,t]} M_s^4\right]^{\frac{1}{2}} + 4\mathbb{E}\left[\sup_{s \in [0,t]} M_s^4\right]^{\frac{1}{2}}\right) (\mathbb{E}[(Q_t^{M,\pi^n})^2])^{\frac{1}{2}}.\end{aligned}$$

□

Lemma 3.24. *The sequence of processes* $\{K_t^{M,\pi^n}\}_{t\in\mathbb{R}_+}$ *is a Cauchy sequence in* $\mathcal{M}_c^2$.

Proof. Since K_t^{M,π^n} is a finite sums of products of $L^4(\Omega)$ random variables $K_t^{M,\pi^n}\in L^2(\Omega)$. We want to prove that K_t^{M,π^n} is a martingale. Consider $s<t\in\mathbb{R}_+$

$$\begin{aligned}\mathbb{E}[K_t^{M,\pi^n}|\mathcal{F}_s] &= \sum_{t_k\leqslant s, t_k\in\pi^n\backslash\{0\}} M_{t_{k-1}}(M_{t_k}-M_{t_{k-1}})+\\ &\quad +\mathbb{E}[M_{t_{\bar{k}-1}}(M_{t_{\bar{k}}}-M_{t_{\bar{k}-1}})|\mathcal{F}_s]+\sum_{s<t_{k+1}}\mathbb{E}[M_{t_{k-1}}(M_{t_k}-M_{t_{k-1}})|\mathcal{F}_s],\end{aligned}$$

where $t_{\bar{k}}$ is such that $t_{\bar{k}-1}<s\leqslant t_{\bar{k}}$. We can use now the Lemma 3.21, obtaining

$$\mathbb{E}[M_{t_{\bar{k}-1}}(M_{t_{\bar{k}}}-M_{t_{\bar{k}-1}})|\mathcal{F}_s]=M_{t_{\bar{k}-1}}\mathbb{E}[M_{t_{\bar{k}}}-M_{t_{\bar{k}-1}}|\mathcal{F}_s]=M_{t_{\bar{k}-1}}(M_s-M_{t_{\bar{k}-1}})$$

$$\mathbb{E}[\mathbb{E}[M_{t_{k-1}}(M_{t_k}-M_{t_{k-1}})|\mathcal{F}_{t_{k-1}}]|\mathcal{F}_s]=\mathbb{E}[M_{t_{k-1}}^2-M_{t_{k-1}}^2|\mathcal{F}_s]=0.$$

We want that for any $m\geqslant n\to+\infty$ and $\ell\in\mathbb{N}$ we have $\mathbb{E}[(K_\ell^{M,\pi^n}-K_\ell^{M,\pi^m})^2]\to 0$. Fix $\ell\in\mathbb{N}$, then if $m\geqslant n$ there is a map $I^{n,m}\colon\pi^{m,t}\to\pi^{n,t}$ associating to any $t_j\in\pi^m$ the maximum point $I^{n,m,t}(t_j)\in\pi^{n,t}$ such that $I^{n,m}(t_j)\leqslant t_j$. We have that

$$\mathbb{E}[(K_\ell^{M,\pi^n}-K_\ell^{M,\pi^m})^2]=\mathbb{E}\left[\left(\sum_{t_j\in\pi^{m,t}}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})(M_{t_j}-M_{t_{j-1}})\right)^2\right]=$$

$$=\sum_{t_j,t_{j'}\in\pi^m}\mathbb{E}[(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})(M_{t_j}-M_{t_{j-1}})(M_{I^{n,m,t}(t_{j'-1})}-M_{t_{j'-1}})(M_{t_{j'}}-M_{t_{j'-1}})]=$$

$$\begin{aligned}&=2\sum_{t_j<t_{j'}\in\pi^m}\mathbb{E}[\mathbb{E}[(M_{I^{n,m,t}(t_{j'-1})}-M_{t_{j'-1}})(M_{t_j}-M_{t_{j-1}})(M_{t_{j'}}-M_{t_{j'-1}})|\mathcal{F}_{t_{j-1}}]\times]\\ &\quad\times(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})]+\sum_{t_j\in\pi^m}\mathbb{E}[(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^2(M_{t_j}-M_{t_{j-1}})^2].\end{aligned}\tag{3.2}$$

If $t_{j-1}<t_j\leqslant t'_{j-1}$, then

$$\begin{aligned}&\mathbb{E}[(M_{I^{n,m,t}(t_{j'-1})}-M_{t_{j'-1}})(M_{t_j}-M_{t_{j-1}})(M_{t_{j'}}-M_{t_{j'-1}})|\mathcal{F}_{t_{j-1}}]=\\ &=\mathbb{E}[\mathbb{E}[(M_{I^{n,m,t}(t_{j'-1})}-M_{t_{j'-1}})(M_{t_j}-M_{t_{j-1}})(M_{t_{j'}}-M_{t_{j'-1}})|\mathcal{F}_{t'_{j-1}}]|\mathcal{F}_{t_{j-1}}]=\\ &=\mathbb{E}[\mathbb{E}[(M_{t_{j'}}-M_{t_{j'-1}})|\mathcal{F}_{t_{j'-1}}](M_{I^{n,m,t}(t_{j'-1})}-M_{t_{j'-1}})(M_{t_j}-M_{t_{j-1}})|\mathcal{F}_{t_j}]=0.\end{aligned}$$

Thus, only the second term in the sum (3.2) is nonzero, and we get

$$\begin{aligned}&\mathbb{E}[(K_\ell^{M,\pi^n}-K_\ell^{M,\pi^m})^2]=\\ &=\mathbb{E}\left[\sum_{t_j\in\pi^n}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^2(M_{t_j}-M_{t_{j-1}})^2\right]\leqslant\\ &\leqslant\mathbb{E}\left[\left(\sup_{t_j\in\pi^{m,t}}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^2\right)Q_t^{M,\pi^m}\right]\leqslant\mathbb{E}\left[\sup_{t_j\in\pi^{m,t}}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^4\right]^{1/2}\times\\ &\quad\times\mathbb{E}[(Q_t^{M,\pi^m})^2]^{\frac{1}{2}}\leqslant C\,\mathbb{E}\left[\sup_{t_j\in\pi^{m,t}}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^4\right]^{1/2}\mathbb{E}[M_t^4].\end{aligned}$$

We have that

$$\mathbb{E}\left[\sup_{t_j\in\pi^{m,t}}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^4\right]\to 0\tag{3.3}$$

as $m\geqslant n\to+\infty$. Indeed, $\sup_{t_j\in\pi^{m,t}}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^4$ converges pointwise to 0 since M_s is a continuous process, and thus, for any $\omega\in\Omega$, $s\mapsto M_s(\omega)$ is a uniformly continuous function when $s\in[0,t]$. Furthermore,

$$\sup_{t_j\in\pi^{m,t}}(M_{I^{n,m,t}(t_{j-1})}-M_{t_{j-1}})^4\leqslant\sup_{s\in[0,t]}M_s^4$$

which, by Doob martingale inequality for $p=4$, and the fact that $M_s \in L^4(\Omega)$ for every $s \in \mathbb{R}_+$, is $\sup_{s\in[0,t]} M_s^4 \in L^1(\Omega)$. Thus, if we apply the Lebesgue dominated convergence theorem to $\mathbb{E}[\sup_{t_j\in\pi^{m,t}} (M_{I^{n,m,t}(t_{j-1})} - M_{t_{j-1}})^4]$ we get the limit. □

Proof of Proposition 3.20. We have that

$$M_t^2 - M_0 - Q_t^{M,\pi^n} = 2K_t^{M,\pi^n}.$$

For Lemma 3.24 the sequence K_t^{M,π^n} converges to some process $\tilde{M} \in \mathcal{M}_c^2$. In particular means that for any $t>0$, $K_t^{M,\pi^n} \to \tilde{M}_t$ in $L^2(\Omega)$ and so $\lim_{n\to+\infty} Q_t^{M,\pi^n}$ exists in $L^2(\Omega)$ and we have

$$\lim_{n\to+\infty} Q_t^{M,\pi^n} = 2\tilde{M} - M_t^2 + M_0^2$$

to some process $\tilde{Q}_t^M$. It is easy to prove (exercise) that $\tilde{Q}_t^M$ is increasing (more precisely non-decreasing) in t, and also $\tilde{Q}_0^M = 0$. Furthermore $M_t^2 - M_0^2 - \tilde{Q}_t^M = \tilde{M}$ is a $\mathcal{M}_c^2$. By the uniqueness of quadratic variation (proved in Proposition 3.18) we have $\tilde{Q}_t^M = [M]_t$. □

3.3.2 Quadratic variation of continuous local martingale

Proof of Theorem 3.19. We can consider a local martingale such that $M_0 = 0$. Indeed in the general case we have $[M]_t = [M - M_0]_t$:

$$(M_t)^2 - [M - M_0]_t = (M_t - M_0 + M_0)^2 - [M - M_0]_t = (M_t - M_0)^2 - [M - M_0]_t + 2M_0 M_t.$$

Since $(M_t - M_0)^2 - [M - M_0]_t$ is a local martingales (by definition of quadratic variation) and $2M_0 M_t$ is a local martingale (being the product of a local martingale and a $\mathcal{F}_0$ measurable random variable) we have that $(M_t)^2 - [M - M_0]_t$ is a local martingale and so $[M]_t = [M - M_0]_t$.

Let M_t be a local martingale with $M_0 = 0$. Then we can define the sequence of increasing stopping times

$$T_n = \inf\{t \geqslant 0, |M_t| \geqslant n\}.$$

We have that $M_t^{T_n}$ is a bounded (and so $\mathcal{M}_c^4$) martingale and so there is the quadratic variation $[M^{T_n}]_t$ and it is such that

$$[M^{T_n}]_t = \lim_{|\pi|\to 0} \sum_{t_i\in\pi\setminus\{0\}} (M_{t_i\wedge t}^{T_n} - M_{t_{i-1}\wedge t}^{T_n})^2 = \lim_{|\pi|\to 0} \sum_{t_i\in\pi\setminus\{0\}} (M_{t_i\wedge t\wedge T_n} - M_{t_{i-1}\wedge t\wedge T_n})^2$$

in $L^2(\Omega)$. So in particular the sequence $\lim_{|\pi|\to 0} \sum_{t_i\in\pi\setminus\{0\}} (M_{t_i\wedge t} - M_{t_{i-1}\wedge t})^2$ converges in probability to some process continuous process $([M^{T_n}]_t)$ on the set $T_n \leqslant t$. Since $T_n \to +\infty$ almost surely, the set $\mathbb{P}(\cup_{n\in\mathbb{N}}\{T_n \leqslant t\}) = 1$. Thus the $[M]_t := \lim_{n\to+\infty} [M^{T_n}]_t = \lim_{|\pi|\to 0} \sum_{t_i\in\pi\setminus\{0\}} (M_{t_i\wedge t} - M_{t_{i-1}\wedge t})^2 < +\infty$ exists almost surely for any $t \in \mathbb{R}_+$. Finally $M_t^2 - [M]_t$ is a local martingale since $(M_t^2 - [M]_t)^{T_n} = (M_t^{T_n})^2 - [M^{T_n}]_t$ are martingales. □

3.3.3 The case of Brownian motion

In this subsection we compute the quadratic variation of Brownian motion.

Theorem 3.25. *Let B_t be an $\mathcal{F}_t$ Brownian motion then*

$$[B]_t = t.$$

Proof. The Brownian motion is a $L^4(\Omega)$ martingale (more generally, it is a $L^p(\Omega)$ martingale for any $1 \leqslant p < +\infty$). This means that for any sequence of increasing partitions $|\pi_n| \to 0$ we have

$$[B]_t = \lim_{|\pi_n|\to 0} \sum_{t_i\in\pi_n^t\setminus\{0\}} (B_{t_i} - B_{t_{i-1}})^2.$$ □

3.3.4 Quadratic covariation

Let M and N be two (local) martingale, then the product MN is in general not a local martingale. For this reason we introduce the following

Definition 3.26. *Let M and N be two continuous local martingales we say that the continuous bounded variation process $[M,N]$ is the* (quadratic) covariation of M and N *if*

$$M_t N_t - [M,N]_t$$

is a local martingale.

Remark 3.27. For the quadratic covariation the following formula holds

$$[M,N]_t = \frac{1}{2}([M+N]_t - [M-N]_t).$$

Indeed,

$$\begin{aligned} M_t N_t - \frac{1}{2}([M+N]_t - [M-N]_t) &= \frac{1}{2}(M_t+N_t)^2 - \frac{1}{2}(M_t-N_t)^2 - \frac{1}{2}([M+N]_t - [M-N]_t) \\ &= \frac{1}{2}[(M_t+N_t)^2 - [M+N]_t] - \frac{1}{2}((M_t-N_t)^2 - [M-N]_t). \end{aligned}$$

We have the following convergence result for the covariation of local continuous martingales.

Theorem 3.28. *Let M,N be two continuous local martingale. If $\{\pi_n\}$ is a sequence of increasing partitions such that $|\pi_n| \to 0$, then*

$$[M,N]_t = \lim_{n\to+\infty} \sum_{t_i\in\pi_n\setminus\{0\}} (M_{t_i\wedge t} - M_{t_{i-1}\wedge t})(N_{t_i\wedge t} - N_{t_{i-1}\wedge t}) \tag{3.4}$$

in probability (or in L^2 if $M,N\in\mathcal{M}_c^4$).

Proof. The result follows from Proposition 3.20 and Theorem 3.19 and from the observation that

$$\begin{aligned} [M,N]_t = \frac{1}{2}([M+N]_t - [M-N]_t) = \lim_{|\pi|\to 0} \frac{1}{2}\sum (M_{t_i} + N_{t_i} - M_{t_{i-1}} - N_{t_{i-1}})^2 + \\ -\frac{1}{2}\lim_{|\pi|\to 0}\sum (M_{t_i} - N_{t_i} - M_{t_{i-1}} + N_{t_{i-1}})^2 = \lim_{|\pi|\to 0}\sum (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}). \end{aligned}$$

□

We have the following useful results.

Proposition 3.29. *Let M, M_1, M_2 and N be some local martingales then*

1. $[M,M]_t = [M]_t$;
2. *for* $\alpha,\beta\in\mathbb{R}$ $[\alpha M_1 + \beta M_2, N]_t = \alpha[M_1,N]_t + \beta[M_2,N]_t$;
3. $[M,N]_t = [N,M]_t$;
4. $[M,N]_t^2 \leqslant [M]_t[N]_t$.

Proof. Exercise *(Hint: use the characterization (3.4)).* □

Chapter 4
Ito Integral and Ito formula

4.1 Integration with respect to continuous martingales

4.1.1 Integration of bounded simple processes and L^2 martingales

Definition 4.1. *Let X_t be an adapted with respect to the filtration $\{\mathcal{F}_t\}$. We say that X_t is a* simple (bounded) process *if there is a $\sigma \in \Pi((0,\infty))$ and a sequence P_{t_n} of $\mathcal{F}_{t_n}$ (bounded) random variables, such that*

$$X_t = \sum_{t_n \in \sigma} P_{t_n} \mathbb{I}_{(t_n, t_{n+1}]}(t).$$

Hereafter we denote by $\mathcal{E}_b$ the set of bounded simple processes.

Definition 4.2. *Let M be an L^2 martingale and X be a simple process define the process Ito integral $X \cdot M$ as the process*

$$(X \cdot M)_t := \int_0^t X_t \mathrm{d}M_t := \sum_{t_n \in \sigma^t} P_{t_n}(M_{t_{n+1} \wedge t} - M_{t_n \wedge t}).$$

Hereafter we identify the integral with respect to the increasing process $[M]_t$ as the Lebesgue integral with respect to the abstract measure $\mu_{[M]}(\mathrm{d}t)$ on $\mathbb{R}_+$ such that

$$\mu_{[M]}((a,b]) = [M]_b - [M]_a, \quad \mu_{[M]}(\{0\}) = 0.$$

If Y_t is an adapted process such that, for almost every $\omega \in \Omega$, the $Y_t(\omega) \in L^1_{\mathrm{loc}}(\mu_{[M](\omega)}(\mathrm{d}t))$ we define

$$\int_0^t Y_s \mathrm{d}[M]_s := \int_0^t Y_s \mu_{[M]}(\mathrm{d}s). \tag{4.1}$$

It is important to note that when Y_t is continuous, since the process $[M]$ is increasing and so it has bounded variation, the integral (4.1) can be interpreted as the Riemann-Stieltjes integral of the function $Y_t(\omega)$ with respect to the function $[M]_t(\omega)$ namely

$$\int_0^t Y_s \mathrm{d}[M]_s = \lim_{|\pi| \to 0} \sum_{t_i \in \pi \setminus \{0\}} Y_{t_{i-1}}(M_{t_i} - M_{t_{i-1}}),$$

almost surely. When instead Y_t is some bounded simple process $Y_t = \sum_{t_n} P'_{t_n} \mathbb{I}_{[t_n, t_{n+1})}(\omega)$ we have

$$\int_0^t Y_s \mathrm{d}[M]_s = \sum_{t_n} P'_{t_n}([M]_{t_{n+1} \wedge t} - [M]_{t_n \wedge t}).$$

The previous formula implies that if Y_t is a bounded simple process with respect to Definition 4.1 then $\int_0^t Y_s \mathrm{d}[M]_s$ is an adapted (continuous) process.

Proposition 4.3. *The process $X\cdot M$ is in $\mathcal{M}_c^2$ and we have*

$$[X\cdot M]_t=\int_0^t X_t^2 \mathrm{d}[M]_t$$

and so

$$\mathbb{E}[((X\cdot M)_t)^2]=\mathbb{E}\left[\int_0^t X_t^2 \mathrm{d}[M]_t\right]. \tag{4.2}$$

Remark 4.4. Equality (4.2) is called *Ito isometry*.

Before proving Proposition 4.3 we prove the following general theorem.

Theorem 4.5. *Let $M\in\mathcal{M}_c^2$ be a continuous $L^2(\Omega)$ martingale then $M_t^2-[M]_t$ is a martingale (and, thus, not only a local martingale).*

Proof. Without loss of generality we can suppose $M_0=0$ (in the other case we take $\tilde{M}_t=M_t-M_0$).

Since both M_t and $[M]_t$ are continuous and being, by definition, $M_t^2-[M]_t$ a local martingale, writing

$$T_n=\inf\{t\geqslant 0, |M_t|, [M]_t\geqslant n\},$$

we have that $(M_t^{T_N})^2-[M]_{t\wedge T_n}$ is a bounded martingale. In particular this means that for any $s<t$

$$\mathbb{E}[(M_t^{T_N})^2-[M]_{t\wedge T_n}|\mathcal{F}_s]=(M_s^{T_N})^2-[M]_{s\wedge T_n}. \tag{4.3}$$

Since $M_t\in\mathcal{M}_c^2$, M_t^2 is a submartingale and so, by Doob stopping time theorem,

$$M_{t\wedge T_n}^2\leqslant \mathbb{E}[M_t^2|\mathcal{F}_{T_n\wedge t}]$$

which implies that the family of random variable $\{M_{t\wedge T_n}^2\}_{n\in\mathbb{N}}$ is uniformly integrable, since the family of random variable $\{\mathbb{E}[M_t|\mathcal{F}_{t\wedge T_n}]\}_{n\in\mathbb{N}}$ is uniformly integrable.

Since T_n is increasing with respect to n, and $[M]_t$ is an increasing process (with respect to t) then $[M]_{t\wedge T_n}$ is increasing with respect to n. Thus by monotone convergence theorem, we have that

$$\mathbb{E}[[M]_t]=\lim_{n\to+\infty}\mathbb{E}[[M]_{t\wedge T_n}]=\lim_{n\to+\infty}\mathbb{E}[(M_t^{T_n})^2]\leqslant\mathbb{E}\left[\lim_{n\to+\infty}(M_t^{T_n})^2\right]\leqslant\mathbb{E}[M_t^2]<+\infty$$

where we used the fact that $(M_t^{T_n})^2$ is uniformly integrable and thus we can exchange the limit with the expectation $\mathbb{E}[\cdot]$. This means that $[M]_t\in L^1(\Omega)$. Furthermore, again by monotone convergence theorem we get

$$\lim_{n\to+\infty}\mathbb{E}[[M]_{t\wedge T_n}|\mathcal{F}_s]=\mathbb{E}\left[\lim_{n\to+\infty}[M]_{t\wedge T_n}\middle|\mathcal{F}_s\right]=\mathbb{E}[[M]_t|\mathcal{F}_s].$$

Finally since $(M_t^{T_n})^2\to M_t^2$ almost surely and $\{(M_t^{T_n})^2\}_{n\in\mathbb{N}}$ is uniformly integrable we get

$$\lim_{n\to+\infty}\mathbb{E}[(M_t^{T_n})^2|\mathcal{F}_s]=\mathbb{E}\left[\lim_{n\to+\infty}(M_t^{T_n})^2\middle|\mathcal{F}_s\right]=\mathbb{E}[M_t^2|\mathcal{F}_s].$$

Thus, taking the limit $n\to+\infty$ in equality (4.3), we obtain

$$\mathbb{E}[M_t^2-[M]_t|\mathcal{F}_s]=M_s^2-[M]_s$$

and, so, that $M_t^2-[M]_t$ is a martingale. $\square$

Proof of Proposition 4.3. Since M is continuous $X\cdot M$ is continuous. Furthermore since both X and M are adapted and $X\cdot M$ is piecewise the product of adapted processes it is also adapted. Finally, since, for any $t\in\mathbb{R}_+$, $X\cdot M_t$ is the finite sum of products of a bounded random variable and a $L^2(\Omega)$ random variable $X\cdot M_t$ is in $L^2(\Omega)$.

We now prove that $X\cdot M$ is a martingale. Consider $t\geqslant s\in\mathbb{R}_+$ then

$$\mathbb{E}[(X\cdot M)_t|\mathcal{F}_s]=\sum_{t_{n+1}\leqslant s}P_{t_n}(M_{t_{n+1}}-M_{t_n})+\mathbb{E}[P_{t_k}(M_{t_{k+1}}-M_{t_k})|\mathcal{F}_s]+\mathbb{E}\left[\sum_{t_n\geqslant s}P_{t_n}(M_{t_{n+1}}-M_{t_n})\middle|\mathcal{F}_s\right]$$

where $t_k \leqslant s \leqslant t_{k+1}$. Then P_{t_k} and M_{t_k} are $\mathcal{F}_{t_k} \subset \mathcal{F}_s$ measurable thus

$$\mathbb{E}[P_{t_k}(M_{t_{k+1}} - M_{t_k})|\mathcal{F}_s] = P_{t_k}(\mathbb{E}[M_{t_{k+1}}|\mathcal{F}_s] - M_{t_k}) = P_{t_k}(M_s - M_{t_k}) = P_{t_k}(M_{s\wedge t_{k+1}} - M_{t_k}).$$

Furthermore we get

$$\begin{aligned}
\mathbb{E}\left[\sum_{t_n \geqslant s} P_{t_n}(M_{t_{n+1}} - M_{t_n})\middle|\mathcal{F}_s\right] &= \mathbb{E}\left[\sum_{t_n \geqslant s} \mathbb{E}[P_{t_n}(M_{t_{n+1}} - M_{t_n})|\mathcal{F}_{t_n}]\middle|\mathcal{F}_s\right]\\
&= \mathbb{E}\left[\sum_{t_n \geqslant s} P_{t_n}(\mathbb{E}[M_{t_{n+1}}|\mathcal{F}_{t_n}] - M_{t_n})\middle|\mathcal{F}_s\right]\\
&= \mathbb{E}\left[\sum_{t_n \geqslant s} P_{t_n}(M_{t_n} - M_{t_n})\middle|\mathcal{F}_s\right] = 0.
\end{aligned}$$

Thus we get that

$$\mathbb{E}[(X\cdot M)_t|\mathcal{F}_s] = \sum_{t_{n+1}\leqslant s} P_{t_n}(M_{t_{n+1}} - M_{t_n}) + P_{t_k}(M_{s\wedge t_{k+1}} - M_{t_k}) = (X\cdot M)_s.$$

We finally prove that $[X\cdot M]_t = \int_0^t X_s^2 \mathrm{d}[M]_s$ using the characterization (3.1), consider $t \in \mathbb{R}_+$, we can consider a sequence $\{\pi_n\}_{n\in\mathbb{N}}$ such that $\sigma \subset \pi_n$, where σ is the partition of $(0, +\infty)$ in the definition of the simple process (i.e. X_t is constant on intervals of the partitions π_n and so

$$X_{t_k} = P_{I^{\pi_n,\sigma}(t_k)}$$

where $I^{\pi_n,\sigma}\colon \pi_n \to \sigma$ is the usal map associating with $t_k \in \pi_n$ the maximum element $t'_{k'} \in \sigma$ such that $t'_{k'} \leqslant t_k$) this means that

$$\begin{aligned}
\lim_{|\pi_n|\to 0} \sum_{t_i\in\pi_n^t\setminus\{0\}} ((X\cdot M)_{t_i} - (X\cdot M)_{t_{i-1}})^2 &= \lim_{|\pi_n|\to 0} \sum_{t_i\in\pi_n^t\setminus\{0\}} X_{t_{i-1}}^2 (M_{t_i} - M_{t_{i-1}})^2 =\\
= \lim_{|\pi_n|\to 0} \sum_{t_r\in\sigma} P_{t_r}^2 &\sum_{s_k\in\pi_n^t\cap[t_r,t_{r+1}]\setminus\{t_r\}} (M_{s_k} - M_{s_{k-1}})^2 =\\
= \sum_{t_r\in\sigma} P_{t_r}^2 \lim_{|\pi_n|\to 0} &\sum_{s_k\in\pi_n^t\cap[t_r,t_{r+1}]\setminus\{t_r\}} (M_{s_k} - M_{s_{k-1}})^2 =\\
= \sum_{t_r\in\sigma} P_{t_r}^2([M]_{t_{r+1}\wedge t} - [M]_{t_r\wedge t}) &= \sum_{t_r\in\sigma} X_{t_r}^2([M]_{t_{r+1}\wedge t} - [M]_{t_r\wedge t}) = \int_0^t X_s \mathrm{d}[M]_s.
\end{aligned}$$

Finally, since $X\cdot M \in \mathcal{M}_c^2$, by Theorem 4.5, $(X\cdot M)_t^2 - [(X\cdot M)]_t = (X\cdot M)_t^2 - \int_0^t X_s^2 \mathrm{d}[M]_s$ is a martingale with $(X\cdot M)_0^2 - [(X\cdot M)]_0 = 0$. This means that

$$\mathbb{E}\left[(X\cdot M)_t^2 - \int_0^t X_s^2 \mathrm{d}[M]_s\right] = \mathbb{E}\left[(X\cdot M)_0^2 - \int_0^0 X_s^2 \mathrm{d}[M]_s\right] = 0. \qquad \square$$

The Ito integral has also the following properties.

Proposition 4.6. *Consider $c\in\mathbb{R}$, $M_1, M_2 \in \mathcal{M}_c^2$, $X_1, X_2 \in \mathcal{E}_b$ and T a stopping time then we have*

1. $(cX_1)\cdot M_1 = c(X_1\cdot M_1)$,
2. $(X_1 + X_2)\cdot M_1 = X_1\cdot M_1 + X_2\cdot M_2$,
3. $[X_1\cdot M_1, X_2\cdot M_2]_t = \int_0^t X_{1,s} X_{2,s} \mathrm{d}[M_1, M_2]_s$,
4. $X_1\cdot M_1^T = (X_1\cdot M)^T$,
5. $[X_1\cdot M_1^T]_t = \int_0^t X_{1,s}^2 \mathrm{d}[M^T]_s = \int_0^{T\wedge t} X_{1,s}^2 \mathrm{d}[M]_s$,
6. $X_2\cdot(X_1\cdot M_1) = (X_1 X_2)\cdot M_1$.

Proof. Exercise. □

4.1.2 Integration of progressive processes and L^2 martingales

Thanks to Ito isometry (4.2) it is possible to extend to more general processes X_t. We introduce the space $L^2_{\text{pro,loc}}(\Omega\times\mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M])$ that is the space of progressive processes Z_t such that for any $t\geqslant 0$ we have

$$\|Z\|^2_{M,[0,t]} = \mathbb{E}\left[\int_0^t Z_s^2 \mathrm{d}[M]_s\right] < +\infty.$$

Hereafter we use the notation

$$L^2_{[0,t]}(M) := L^2_{\text{pro}}(\Omega\times[0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M]), t\geqslant 0$$

$$L^2(M) := L^2_{\text{pro,loc}}(\Omega\times\mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M]) = \bigcap_{t>0} L^2_{\text{pro}}(\Omega\times[0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M]).$$

Exercise 4.1. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, and consider the subset $\mathcal{P}$ of $\mathcal{P}\subset\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)$ defined as

$$A\in\mathcal{P} \Longleftrightarrow \forall t\geqslant 0\, (A\cap(\Omega\times[0,t]))\in\mathcal{F}_t\otimes\mathcal{B}([0,t]).$$

Prove that $\mathcal{P}$ is a σ-algebra, and that the process $X.(\cdot)\colon \mathbb{R}_+\times\Omega\to\mathbb{R}$ is progressive if and only if X is $\mathcal{P}$ measurable.

Remark 4.7. The space $L^2(M)$ is a complete metric space with distance given by (for example by)

$$d(H,K) = \sum_{\ell\in\mathbb{N}} 2^{-\ell}\left\{\left(\mathbb{E}\left[\int_0^\ell |H_s-K_s|^2\mathrm{d}[M]_s\right]\right)^{\frac{1}{2}}\wedge 1\right\},\quad K,H\in L^2(M).$$

For any $t\geqslant 0$ $L^2_{[0,t]}(M)$ is a Hilbert space with scalar product given by

$$(K,H)_{L^2_{[0,t]}(M)} = \mathbb{E}\left[\int_0^t K_sH_s\mathrm{d}[M]_s\right],\quad K,H\in L^2_{[0,t]}(M).$$

Finally a sequence $K^n\to K$ in $L^2(M)$ with respect the metric above, if and only if, for any $t\geqslant 0$, $K^n\to K$ converges to K in $L^2_{[0,t]}(M)$.

Exercise 4.2. Using Exercise 4.1, prove the assertions in Remark 4.7.

Indeed, the following proposition holds.

Proposition 4.8. *Consider $Z\in L^2_{\text{pro,loc}}(\Omega\times\mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M])$. Then there is a sequence $\{X^n\}_{n\in\mathbb{N}}$ of simple processes such that, for any $t\in\mathbb{R}_+$,*

$$\mathbb{E}\left[\int_0^t (Z_s-X_s^n)^2\mathrm{d}[M]_s\right]\to 0,\quad n\to+\infty.$$

Proof. By Remark 4.7, the thesis of Proposition 4.8 is equivalent to prove that, for any $t>0$, $\mathcal{E}_b$ is dense in $L^2_{\text{pro}}(\Omega\times[0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M])$. Since $L^2_{\text{pro}}(\Omega\times[0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M])$ is a Hilbert space, the density of the subspace $\mathcal{E}_b\subset L^2_{\text{pro}}(\Omega\times[0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M])$ is equivalent to the fact that $\mathcal{E}_b^\perp=\{0\}$, where the $\mathcal{E}_b^\perp\subset L^2_{\text{pro}}(\Omega\times[0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M])$ is the orthogonal subspace of $\mathcal{E}_b$ namely $K_s\in\mathcal{E}_b^\perp$ if and only if for any $X_s\in\mathcal{E}_b$ we have

$$\mathbb{E}\left[\int_0^t X_sK_s\mathrm{d}[M]_s\right]=0.$$

So consider $K\in\mathcal{E}_b^\perp$ and define the process

$$Y_s=\int_0^s K_r\mathrm{d}[M]_r.$$

The process Y_s is an adapted $L^1(\Omega)$ process (it is adapted since K_r is progressive and $[M]_r$ is continuous and adapted and thus it is progressive). Since $[M]_s$ is continuous Y_s is continuous and since $[M]_s$ is of bounded variation and $K_s \in L^2([0,t],\mathrm{d}[M]) \subset L^1([0,t],\mathrm{d}[M])$ almost surely Y_s is of bounded variation almost surely. Consider $s_1 < s_2 \leqslant t$ and $F \in \mathcal{F}_{s_1}$, then $X_s = \mathbb{I}_F(\omega)\mathbb{I}_{[s_1,s_2]}(s) \in \mathcal{E}_b$. This means that

$$0 = \mathbb{E}\left[\int_0^t X_s K_s \mathrm{d}[M]_s\right] = \mathbb{E}\left[\mathbb{I}_F \int_{s_1}^{s_2} K_s \mathrm{d}[M_s]\right] = \mathbb{E}[\mathbb{I}_F(Y_{s_2} - Y_{s_1})],$$

and thus

$$\mathbb{E}[\mathbb{I}_F Y_{s_2}] = \mathbb{E}[\mathbb{I}_F Y_{s_1}].$$

Since the last equality holds for a generic $F \in \mathcal{F}_{s_1}$, by definition of conditional probability we get

$$\mathbb{E}[Y_{s_1}|\mathcal{F}_{s_2}] = Y_{s_2}.$$

This means that Y_s is a continuous bounded variation martingale, and thus, by Theorem 3.16, $Y_s = Y_0 = 0$ almost surely. Since $[M]_s$ is an increasing continuous process, this means that K_s must be zero with respect to a set of measure 0 of the measure $\mathrm{d}\mathbb{P}\mathrm{d}[M]$, and by definition of $L^2_{\mathrm{pro}}(\Omega \times [0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M])$, $K = 0$ as an element of $L^2_{\mathrm{pro}}(\Omega \times [0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M])$. This means that $\mathcal{E}_b^{\perp} = \{0\}$ and so $\mathcal{E}_b$ is dense in $L^2_{\mathrm{pro}}(\Omega \times [0,t], \mathrm{d}\mathbb{P}\mathrm{d}[M])$ and so on $L^2_{\mathrm{pro,loc}}(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M])$. □

Definition 4.9. *Consider* $Z \in L^2_{\mathrm{pro,loc}}(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M])$ *and* $M \in \mathcal{M}_c^2$ *we define the* Ito integral $Z \cdot M$ *as the limit*

$$Z \cdot M = \lim_{n \to 0} X^n \cdot M$$

where $X^n \in \mathcal{E}_b$ *is some sequence of bounded simple processes such that for any* $t > 0$ $\mathbb{E}[\int_0^t (Z_s - X_s^n)^2 \mathrm{d}[M]_s] \to 0$ *as* $n \to +\infty$.

Theorem 4.10. *If* $Z \in L^2_{\mathrm{pro,loc}}(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M])$ *the Ito integral* $Z \cdot M$ *is well defined (i.e. it exists and it is unique up to indistiguishability), furthermore* $Z \cdot M \in \mathcal{M}_c^2$

$$[Z \cdot M]_t = \int_0^t Z_s^2 \mathrm{d}[M]_s$$

and

$$\mathbb{E}[(Z \cdot M)_t^2] = \mathbb{E}\left[\int_0^t Z_s^2 \mathrm{d}[M]_s\right].$$

Proof. By Proposition 4.8, there is a sequence $X^n \in \mathcal{E}_b$ such that $\mathbb{E}[\int_0^t (X_s^n - Z_s)^2 \mathrm{d}[M]_s] \to 0$. In particular this means that the sequence $X^n \cdot M$ is a Cauchy sequence in $\mathcal{M}_c^2$ indeed

$$\begin{aligned} d_{\mathcal{M}_c^2}(X^n \cdot M, X^m \cdot M) &= \sum_{\ell \in \mathbb{R}} 2^{-\ell}(\mathbb{E}[|(X^n \cdot M)_\ell - (X^m \cdot M)_\ell|^2] \wedge 1) \\ &= \sum_{\ell \in \mathbb{R}} 2^{-\ell}\left(\mathbb{E}\left[\int_0^\ell (X_s^n - X_s^m)^2 \mathrm{d}[M]_s\right] \wedge 1\right) \\ &= \sum_{\ell \in \mathbb{R}} 2^{-\ell}\left(\left(\mathbb{E}\left[\int_0^\ell (X_s^n - Z)^2 \mathrm{d}[M]_s\right] + \mathbb{E}\left[\int_0^\ell (X_s^m - Z)^2 \mathrm{d}[M]_s\right]\right) \wedge 1\right) \end{aligned}$$

which converges to 0 when $n, m \to +\infty$. This implies that $X^n \cdot M$ converges to some $Z \cdot M \in \mathcal{M}_c^2$. Suppose that $X^{n\prime}$ is such that $\mathbb{E}[\int_0^t |X_s^{n\prime} - Z_s|^2 \mathrm{d}[M]_s] \to 0$ then $X^n \cdot M - X^{n\prime} \cdot M$ converges to 0 in $\mathcal{M}_c^2$. This implies that $Z \cdot M$ is uniquely defined up to indistiguishability.

Since $X^n \cdot M$ converges to $Z \cdot M$ in $\mathcal{M}_c^2$, it means that for any $t \in \mathbb{R}_+$ the random variable $(X^n \cdot M)_t^2$ converges to $(Z \cdot M)_t^2$ in $L^1(\Omega)$. Furthermore we have that

$$\begin{aligned} \mathbb{E}\left[\left|\int_0^t (X_s^n)^2 \mathrm{d}[M]_s - \int_0^t Z_s^2 \mathrm{d}[M]_s\right|\right] &\leqslant \mathbb{E}\left[\int_0^t |(X_s^n)^2 - Z_s^2| \mathrm{d}[M]_s\right] \\ &\leqslant \left(\mathbb{E}\left[\int_0^t (X_s^n + Z_s)^2 \mathrm{d}[M]_s\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^t (X_s^n - Z_s)^2 \mathrm{d}[M]_s\right]\right)^{\frac{1}{2}} \end{aligned}$$

and so $\int_0^t (X_s^n)^2 \mathrm{d}[M]_s$ converges to $\int_0^t Z_s^2 \mathrm{d}[M]_s$ in $L^1(\Omega)$. Furthermore, by what we say before, $(X^n \cdot M)_t$ converges to $(Z \cdot M)_t$ in $L^2(\Omega)$ and so $(X^n \cdot M)_t^2$ converges to $(Z \cdot M)_t$ in $L^1(\Omega)$. So let $t \geqslant s$ we have that

$$\begin{aligned}\mathbb{E}\left[(Z \cdot M)_t^2 - \int_0^t Z_\tau^2 \mathrm{d}[M]_\tau \middle| \mathcal{F}_s\right] &= \lim_{n \to +\infty} \mathbb{E}\left[(X^n \cdot M)_t - \int_0^t (X_\tau^n)^2 \mathrm{d}[M]_\tau \middle| \mathcal{F}_s\right] \\ &= \lim_{n \to +\infty} \left((X^n \cdot M)_s - \int_0^s (X_\tau^n)^2 \mathrm{d}[M]_\tau\right) \\ &= (Z \cdot M)_s^2 - \int_0^s Z_\tau^2 \mathrm{d}[M]_\tau \end{aligned} \tag{4.4}$$

where we used that, by Theorem 4.5, $\mathbb{E}[(X^n \cdot M)_t - \int_0^t (X_\tau^n)^2 \mathrm{d}[M]_\tau | \mathcal{F}_s] = (X^n \cdot M)_s - \int_0^s (X_\tau^n)^2 \mathrm{d}[M]_\tau$. Equality (4.10) proves that $(Z \cdot M)_t - \int_0^t Z_\tau^2 \mathrm{d}[M]_\tau$ is a martingale and by uniqueness of quadratic variation

$$[Z \cdot M]_t = \int_0^t Z_\tau^2 \mathrm{d}[M]_\tau.$$

Finally since $Z \cdot M \in \mathcal{M}_c^2$, again by Theorem 4.5,

$$\mathbb{E}[(Z \cdot M)_t^2] = \mathbb{E}\left[\int_0^t Z_s^2 \mathrm{d}[M]_s\right]. \qquad \square$$

The Ito integral $Z \cdot M$ satisfies some useful properties.

Proposition 4.11. *Consider $c \in \mathbb{R}$, $M_1, M_2 \in \mathcal{M}_c^2$, $Z_1, Z_2, \widetilde{Z}_1, \widetilde{Z}_2$ progressive processes such that $Z_1, Z_2 \in L_{\mathrm{loc}}^2(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M_1])$, $Z_1 \in L^2(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M_2])$ and $\tilde{Z}_1, \tilde{Z}_2 \in L^4(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M_1])$ and T a stopping time then we have*

1. $(cZ_1) \cdot M_1 = c(Z_1 \cdot M_1)$;
2. $(Z_1 + Z_2) \cdot M_1 = Z_1 \cdot M_1 + Z_2 \cdot M_2$;
3. $Z_1 \cdot (M_1 + M_2) = Z_1 \cdot M_1 + Z_1 \cdot M_2$;
4. $[Z_1 \cdot M_1, Z_2 \cdot M_2]_t = \int_0^t Z_{1,s} Z_{2,s} \mathrm{d}[M_1, M_2]_s$,
5. $Z_1 \cdot M^T = (Z_1 \cdot M)^T$,
6. $[Z_1 \cdot M^T]_t = \int_0^t Z_{1,s}^2 \mathrm{d}[M^T]_s = \int_0^{T \wedge t} Z_{1,s}^2 \mathrm{d}[M]_s$,
7. $\tilde{Z}_2 \cdot (\tilde{Z}_1 \cdot M_1) = (\tilde{Z}_2 \tilde{Z}_1) \cdot M_1$.

Proof. Exercise. $\square$

We want to conclude this section with an important approximation theorem when the process Z_t is a continuous adapted process.

Theorem 4.12. *Let Z_t be a continuous bounded adapted process such that, for any $t \in \mathbb{R}_+$, $\mathbb{E}[\int_0^t Z_s^2 \mathrm{d}[M]_s] < +\infty$ then if $\{\pi_n\}_{n \in \mathbb{N}}$ is a sequence of increasing partitions such that $|\pi_n| \to 0$ we have*

$$(Z \cdot M)_t = \lim_{n \to 0} \sum_{t_i \in \pi_n^t \setminus \{0\}} Z_{t_{i-1}} (M_{t_i} - M_{t_{i-1}}) \tag{4.5}$$

in $L^2(\Omega)$.

Proof. We write

$$Z_t^{\pi_n} = \sum_{t_i \in \pi_n \setminus \{0\}} Z_{t_{i-1}} \mathbb{I}_{(t_{i-1}, t_i]}(t).$$

If we prove that

$$\mathbb{E}\left[\int_0^t |Z_t^{\pi_n} - Z_t|^2 \mathrm{d}[M]_s\right] \to 0$$

the theorem is proved. We have that

$$\mathbb{E}\left[\int_0^t |Z_t^{\pi_n} - Z_t|^2 \mathrm{d}[M]_s\right] \leqslant \mathbb{E}\left[\max_{t_i \in \pi_n^t \setminus \{0\}} \left(\sup_{s \in [t_{i-1}, t_i]} |Z_{t_{i-1}} - Z_s|\right)[M]_t\right].$$

On the other hand, since $|Z| \leqslant K$ for some constant K,

$$\max_{t_i \in \pi_n \setminus \{0\}} \left(\sup_{t \in [t_{i-1}, t_i]} |Z_{t_{i-1}} - Z_t|\right)[M]_t \leqslant 2K[M]_t$$

which is an $L^1(\Omega)$ random variable (since $M \in \mathcal{M}_c^2$). Furthermore

$$\max_{t_i \in \pi_n \setminus \{0\}} \left(\sup_{s \in [t_{i-1}, t_i]} |Z_{t_{i-1}} - Z_s|\right) \to 0, \quad n \to +\infty$$

since $t \mapsto Z_t$ is continuous on $\mathbb{R}_+$ and so uniformly continuous on $[0, t]$. By Lebesgue dominated convergence theorem we get the thesis. □

Remark 4.13. The boundedness of Z can be replaced with the condition that, for any $t \in \mathbb{R}_+$, $\sup_{s \in [0,t]} |Z_s| \in L^p(\Omega)$ and $[M]_t \in L^q(\Omega)$ for $\frac{1}{p} + \frac{1}{q} \leqslant 1$.

More generally, if M is a local martingale and Z is a continuous process, then the convergence in (4.5) holds in probability.

4.1.3 The Brownian motion case

The integration with respect Brownian motion is a special case of integration with respect $\mathcal{M}_c^2$ martingales introduced above. In this case the space of progressive processes Z_t that can be integrated must satisfy the condition

$$\mathbb{E}\left[\int_0^t Z_s^2 \mathrm{d}s\right] < +\infty;$$

in other words they must be progressive function in the space

$$L^2_{\mathrm{loc}}(\Omega \times \mathbb{R}_+, \mathbb{P} \otimes \mathrm{d}t).$$

The Ito integral $(Z \cdot B)_\cdot = \int_0^\cdot Z_s \mathrm{d}B_s$ is an $\mathcal{M}_c^2$ martingale and we have

$$\left[\int_0^\cdot Z_s \mathrm{d}B_s\right]_t = \int_0^t Z_s^2 \mathrm{d}s, \quad \mathbb{E}\left[\left(\int_0^t Z_s \mathrm{d}B_s\right)^2\right] = \mathbb{E}\left[\int_0^t Z_s^2 \mathrm{d}s\right]$$

the second equality is usually called Ito isometry.

4.1.4 Integration with respect to local martingale

In this section let M_t be a continuous local martingale such that $M_0 = 0$. Consider a progressive process such that for any $t \in \mathbb{R}_+$

$$\int_0^t X_s^2 \mathrm{d}[M]_s < +\infty$$

almost surely.

Theorem 4.14. *Let M_t a local martingale and X_t a progressive process such that $\int_0^t X_s^2 \mathrm{d}[M]_s < +\infty$, then there is a unique (up to indistiguishability) process $X \cdot M$ such that there is a sequence of increasing stopping times $T_n \to +\infty$ such that $M^{T_n} \in \mathcal{M}_c^2$ and $X^{T_n} \in L^2_{\mathrm{loc}}(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P}\mathrm{d}[M]^{T_n})$ such that*

$$(X \cdot M)^{T_n} = (X^{T_n} \cdot M^{T_n}).$$

Proof. We can choose the sequence of stopping times T_n in the following way

$$\begin{aligned} T_n^{(1)} &= \inf\{t \geqslant 0, |M_t| \geqslant n\} \\ T_n^{(2)} &= \inf\left\{t \geqslant 0, \int_0^t X_s^2 \mathrm{d}[M]_s \geqslant n\right\} \end{aligned}$$

and so $T_n = T_n^{(1)} \wedge T_n^{(2)}$. Since M^{T_n} is a bounded martingale it is in $\mathcal{M}_c^2$. Furthermore, by Proposition 4.11, we have

$$\mathbb{E}\left[\int_0^t X_s^{T_n} \mathrm{d}[M^{T_n}]_s\right] = \mathbb{E}\left[\int_0^{t\wedge T_n} X_s^{T_n} \mathrm{d}[M]_s\right] \leqslant n$$

this means that there is a unique $\mathcal{M}_c^2$ martingale defined as $X^{T_n} \cdot M^{T_n}$. Since $T_n \to +\infty$ as $n \to +\infty$ the process $X \cdot M$ is well defined. Finally by Proposition 4.11 and the uniqueness of Ito integral for L^2 martingales the process $X \cdot M$ is unique. □

4.2 Ito Formula continuous semimartingales

4.2.1 One dimensional Ito formula

Definition 4.15. *Let X_t be a continuous adapted process. We say that X_t is an L^p-*continuous semimartingale *if there are a bounded variation process A_t such that, for any $t \in \mathbb{R}_+$, $V_t(A.) \in L^p(\Omega)$ (i.e. the variation of the process A_t is an L^p random variable), and an martingale M in $L^p(\Omega)$ such that*

$$X_t = A_t + M_t. \tag{4.6}$$

An adapted process X_t is called a semimartingale *if the decomposition (4.6) holds, where A_t is a continuous bounded variation process and M is a local martingale.*

Remark 4.16. Thanks to Theorem 3.16, the decomposition (4.6) of a semimartingale is unique; i.e. if $X_t = A_t^1 + M_t^1$ and $X_t = A_t^2 + M_t^2$ for some continuous bounded variation processes A^1, A^2 and some continuous local martingales M^1, M^2 then $A^1 = A^2$ almost surely and $M^1 = M^2$ almost surely.

Hereafter if Z_t is a predictable process such that the Riemann-Stieltjes integral $\int_0^t Z_s \mathrm{d}A_s$ exists and such that $\int_0^t Z_s^2 \mathrm{d}[M]_s < +\infty$ almost surely we define

$$\int_0^t Z_s \mathrm{d}X_s := \int_0^t Z_s \mathrm{d}A_s + \int_0^t Z_s \mathrm{d}M_s$$

where the first is a Riemann-Stieltjes integral and the second is the Ito integral with respect to the local martingale M_s. Furthermore we defined the quadratic variation of the seminartingale X_t as

$$[X]_t := [M]_t.$$

Finally if $X^1 = A^1 + M^1, X^2 = A^2 + M^2$ are two continuous semimartingales we define the quadratic covariation as

$$[X^1, X^2]_t := [M^1, M^2]_t.$$

Theorem 4.17. *Let $f \in C^2(\mathbb{R}, \mathbb{R})$ be a bounded function with bounded first and second derivatives, then if X_t is an L^4 semimartingale then $f(X_t)$ is an L^2 semimartingale and we have*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) \mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s) \mathrm{d}[X]_s \\ &= f(X_0) + \int_0^t f'(X_s) \mathrm{d}A_s + \int_0^t f'(X_s) \mathrm{d}M_s + \frac{1}{2}\int_0^t f''(X_s) \mathrm{d}[M]_s. \end{aligned} \tag{4.7}$$

Lemma 4.18. *Let M_t be a $\mathcal{M}_c^4$ martingale then for any partition $\pi \in \Pi([0,t])$ we have*

$$\mathbb{E}\left[\left(\sum_{t_i \in \pi \setminus \{0\}} ((M_{t_i} - M_{t_{i-1}})^2 - [M]_{t_i} + [M]_{t_{i-1}})\right)^2\right]^{1/2} \leqslant 3\left(\left[\sup_{s \in [0,t]} |M_s|^4\right]^{1/2} + \mathbb{E}[[M]_t^2]^{1/2}\right) \times$$

$$\times \left(\mathbb{E}\left[\sup_{t_i \in \pi \setminus \{0\}} (M_{t_i} - M_{t_{i-1}})^4\right]^{1/2} + \mathbb{E}\left[\sup_{t_i \in \pi \setminus \{0\}} ([M]_{t_i} - [M]_{t_{i-1}})^2\right]^{1/2}\right).$$

Proof. We have that

$$\mathbb{E}\left[\left(\sum_{t_i \in \pi \setminus \{0\}} ((M_{t_i} - M_{t_{i-1}})^2 - [M]_{t_i} + [M]_{t_{i-1}})\right)^2\right] =$$

$$= \sum_{t_i, t_{i'} \in \pi \setminus \{0\}} \mathbb{E}[((M_{t_i} - M_{t_{i-1}})^2 - [M]_{t_i} + [M]_{t_{i-1}})((M_{t_{i'}} - M_{t_{i'-1}})^2 - [M]_{t_{i'}} + [M]_{t_{i'-1}})].$$

If $t_i \neq t_{i'}$, we can assume that $t_i > t_{i'}$ then

$$\begin{aligned}
&\mathbb{E}[((M_{t_i} - M_{t_{i-1}})^2 - [M]_{t_i} + [M]_{t_{i-1}})((M_{t_{i'}} - M_{t_{i'-1}})^2 - [M]_{t_{i'}} + [M]_{t_{i'-1}})] = \\
&= \mathbb{E}[\mathbb{E}[(M_{t_i} - M_{t_{i-1}})^2 - [M]_{t_i} + [M]_{t_{i-1}} | \mathcal{F}_{t_{i-1}}]((M_{t_{i'}} - M_{t_{i'-1}})^2 - [M]_{t_{i'}} + [M]_{t_{i'-1}})] = \\
&= \mathbb{E}[(\mathbb{E}[M_{t_i}^2 - [M]_{t_i} | \mathcal{F}_{t_i}] - M_{t_{i-1}}^2 + [M]_{t_{i-1}})((M_{t_{i'}} - M_{t_{i'-1}})^2 - [M]_{t_{i'}} + [M]_{t_{i'-1}})] = \\
&= \mathbb{E}[(M_{t_{i-1}}^2 - [M]_{t_{i-1}} - M_{t_{i-1}}^2 + [M]_{t_{i-1}})((M_{t_{i'}} - M_{t_{i'-1}})^2 - [M]_{t_{i'}} + [M]_{t_{i'-1}})] = 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{t_i \in \pi \setminus \{0\}} ((M_{t_i} - M_{t_{i-1}})^2 - [M]_{t_i} + [M]_{t_{i-1}})\right)^2\right] = \\
&= \mathbb{E}\left[\sum_{t_i \in \pi \setminus \{0\}} ((M_{t_i} - M_{t_{i-1}})^2 - [M]_{t_i} + [M]_{t_{i-1}})^2\right] \\
&\leqslant \mathbb{E}\left[\left\{\sup_{t_i \in \pi \setminus \{0\}} (M_{t_i} - M_{t_{i-1}})^2\right\}\left(\sum_{t_i \in \pi \setminus \{0\}} (M_{t_i} - M_{t_{i-1}})^2\right)\right] + \\
&+ 2\mathbb{E}\left[\left\{\sup_{t_i \in \pi \setminus \{0\}} (M_{t_i} - M_{t_{i-1}})^2\right\}[M]_t\right] + \mathbb{E}\left[\left\{\sup_{t_i \in \pi \setminus \{0\}} ([M]_{t_i} - [M]_{t_{i-1}})\right\}[M]_t\right].
\end{aligned}$$

The thesis follows applying Holder inequality and Lemma 3.23. □

Proof. Let $\{\pi_n\}_{n \in \mathbb{N}} \subset \prod([0,t])$ be a increasing sequence of partition such that $|\pi_n| \to 0$ then, by Lagrange theorem, we have

$$\begin{aligned}
f(X_t) &= f(X_0) + \sum_{t_i \in \pi_n \setminus \{0\}} (f(X_{t_i}) - f(X_{t_{i-1}})) \\
&= f(X_0) + \sum_{t_i \in \pi_n \setminus \{0\}} f'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + \frac{1}{2} \sum_{t_i \in \pi_n \setminus \{0\}} f''(\xi_{X_{t_i}, X_{t_{i-1}}})(X_{t_i} - X_{t_{i-1}})^2
\end{aligned}$$

where $\xi_{X_{t_i}, X_{t_{i-1}}}$ is some point between X_{t_i} and $X_{t_{i-1}}$.

On the other hand

$$\begin{aligned}
\sum_{t_i \in \pi_n \setminus \{0\}} f'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) &= \sum_{t_i \in \pi_n \setminus \{0\}} f'(X_{t_{i-1}})(A_{t_i} - A_{t_{i-1}}) + \\
&+ \sum_{t_i \in \pi_n \setminus \{0\}} f'(X_{t_{i-1}})(M_{t_i} - M_{t_{i-1}}).
\end{aligned}$$

On the other hand, by Theorem 3.12,

$$\sum_{t_i \in \pi_n \setminus \{0\}} f'(X_{t_{i-1}})(A_{t_i} - A_{t_{i-1}}) \to \int_0^t f'(X_s) \mathrm{d}A_s$$

almost surely, since $t\mapsto f'(X_t)$ is a continuous map and $t\mapsto A_t$ has bounded variation. Furthermore since $V_t(A)\in L^4(\Omega)$, the previous convergence is also in $L^4(\Omega)$ and so

$$V_t\left(\int_0^\cdot f'(X_s)\mathrm{d}A_s\right)\leqslant \|f'\|_{L^\infty}V_t(A),$$

which is in $L^4(\Omega)\subset L^2(\Omega)$. Furthermore by Theorem 4.12

$$\sum_{t_i\in\pi_n\setminus\{0\}} f'(X_{t_{i-1}})(M_{t_i}-M_{t_{i-1}})\to\int_0^t f'(X_s)\mathrm{d}M_s$$

in $L^2(\Omega)$ (actually as a martingale in $\mathcal{M}_c^2$).

For the remaining term $\sum_{t_i\in\pi\setminus\{0\}} f''(\xi_{X_{t_i},X_{t_{i-1}}})(X_{t_i}-X_{t_{i-1}})^2$, we want to prove that it converges to $\frac{1}{2}\int_0^t f''(X_t)\mathrm{d}[M]_s$ in $L^2(\Omega)$. Indeed we have

$$\begin{aligned}&\left|\sum_{t_i\in\pi\setminus\{0\}} f''(\xi_{X_{t_i},X_{t_{i-1}}})(X_{t_i}-X_{t_{i-1}})^2-\int_0^t f''(X_t)\mathrm{d}[M]_s\right|=\\ =&\frac{1}{2}\left|\sum_{t_i\in\pi\setminus\{0\}} f''(\xi_{X_{t_i},X_{t_{i-1}}})((M_{t_i}-M_{t_{i-1}})^2-[M]_{t_i}+[M]_{t_{i-1}})\right|+\\ &+\sum_{t_i\in\pi\setminus\{0\}}|f''(\xi_{X_{t_i},X_{t_{i-1}}})||M_{t_i}-M_{t_{i-1}}||A_{t_i}-A_{t_{i-1}}|+\frac{1}{2}\sum_{t_i\in\pi\setminus\{0\}}\int_{t_{i-1}}^{t_i}|f''(\xi_{X_{t_i},X_{t_{i-1}}})-f''(X_s)|\mathrm{d}[M]_s.\end{aligned}$$

By Lemma 4.18, we have

$$\begin{aligned}&\mathbb{E}\left[\left|\sum_{t_i\in\pi_n\setminus\{0\}} f''(\xi_{X_{t_i},X_{t_{i-1}}})((M_{t_i}-M_{t_{i-1}})^2-[M]_{t_i}+[M]_{t_{i-1}})\right|^2\right]\leqslant\\ &\leqslant 3\|f''\|_{L^\infty}3\left(\left[\sup_{s\in[0,t]}|M_s|^4\right]^{1/2}+\mathbb{E}[[M]_t^2]^{1/2}\right)\times\\ &\times\left(\mathbb{E}\left[\sup_{t_i\in\pi_n\setminus\{0\}}(M_{t_i}-M_{t_{i-1}})^4\right]^{1/2}+\mathbb{E}\left[\sup_{t_i\in\pi_n\setminus\{0\}}([M]_{t_i}-[M]_{t_{i-1}})^2\right]^{1/2}\right)\end{aligned}$$

The last factor converges to 0, as $|\pi_n|\to 0$, since $\sup_{t_i\in\pi_n\setminus\{0\}}(M_{t_i}-M_{t_{i-1}})^4\to 0$, $\sup_{t_i\in\pi_n\setminus\{0\}}([M]_{t_i}-[M]_{t_{i-1}})^2\to 0$ almost surely, by the uniform continuity of M_s and $[M]_s$ on $[0,t]$, and by Lebesgue dominated convergence theorem, which can be used because $\sup_{t_i\in\pi_n\setminus\{0\}}(M_{t_i}-M_{t_{i-1}})^4\leqslant\sup_{s\in[0,t]}M_s^4\in L^1(\Omega)$ and $\sup_{t_i\in\pi_n\setminus\{0\}}([M]_{t_i}-[M]_{t_{i-1}})^2\leqslant[M]_t^2\in L^1(\Omega)$. Furthermore we get

$$\begin{aligned}&\mathbb{E}\left[\left|\sum_{t_i\in\pi\setminus\{0\}}|f''(\xi_{X_{t_i},X_{t_{i-1}}})||M_{t_i}-M_{t_{i-1}}||A_{t_i}-A_{t_{i-1}}|\right|^2\right]\leqslant\\ \leqslant&\|f''\|_{L^\infty}\mathbb{E}\left[\sup_{t_i\in\pi_n\setminus\{0\}}|M_{t_i}-M_{t_{i-1}}|^2V_t(A)^2\right]\leqslant\|f''\|_{L^\infty}\mathbb{E}\left[\sup_{t_i\in\pi_n\setminus\{0\}}|M_{t_i}-M_{t_{i-1}}|^4\right]^{1/2}\mathbb{E}[V_t(A)^4]^{\frac{1}{2}}\end{aligned}$$

which converges to 0 since $\mathbb{E}[\sup_{t_i\in\pi_n\setminus\{0\}}|M_{t_i}-M_{t_{i-1}}|^4]^{1/2}\to 0$ as $|\pi_n|\to 0$ as shown above. Finally

$$\begin{aligned}&\mathbb{E}\left[\left|\int_{t_{i-1}}^{t_i}\left(\sum_{t_i\in\pi\setminus\{0\}}|f''(\xi_{X_{t_i},X_{t_{i-1}}})-f''(X_s)|\right)\mathrm{d}[M]_s\right|^2\right]\leqslant\\ \leqslant&\mathbb{E}\left[[M]_t^2\left\{\max_{t_i\in\pi_n\setminus\{0\}}\sup_{s\in[t_i,t_{i-1}]}[|f''(\xi_{X_{t_i},X_{t_{i-1}}})-f''(X_s)|]\right\}\right]\leqslant\\ \leqslant&\mathbb{E}\left[[M]_t^2\left\{\max_{t_i\in\pi_n\setminus\{0\}}\sup_{s\in[t_i,t_{i-1}],k\in[0,1]}[|f''(X_{t_i}+k(X_{t_{i-1}}+X_{t_i}))-f''(X_s)|]\right\}\right]\end{aligned}$$

The last term goes to 0, since by uniform continuity of X_s on $[0,t]$ and the continuity of f'' (and thus the uniform continuity of f'' on compact subsets of $\mathbb{R}$) we have

$$\max_{t_i\in\pi_n\backslash\{0\}}\sup_{s\in[t_i,t_{i-1}],k\in[0,1]}[|f''(X_{t_i}+k(X_{t_{i-1}}+X_{t_i}))-f''(X_s)|]\to 0$$

almost surely. Furthermore since

$$[M]_t^2\left\{\max_{t_i\in\pi_n\backslash\{0\}}\sup_{s\in[t_i,t_{i-1}],k\in[0,1]}[|f''(X_{t_i}+k(X_{t_{i-1}}+X_{t_i}))-f''(X_s)|]\right\}\leqslant 2\|f''\|_{L^\infty}[M]_t^2\in L^1(\Omega),$$

we can apply Lebesgue dominated convergence theorem, obtaining the thesis. □

Theorem 4.17 can be generalize to the case of “generic” semimartingale.

Theorem 4.19. *Let X_t be a continuous semimartingale and let $f\in C^2(\mathbb{R},\mathbb{R})$. Then $f(X_t)$ is a semimartingale and we have*

$$f(X_t)=f(X_0)+\int_0^t f'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^t f''(X_s)\mathrm{d}[X]_s$$

Proof. As usual there is a sequence of stopping time $T_n\to+\infty$ such that $|A_t^{T_n}|,|M_t^{T_n}|,|X_t^{T_n}|\leqslant n$ on the set $\{|X_0|\leqslant n\}$ (recall that since $\{|X_0|\leqslant n\}\in\mathcal{F}_0$ the process $M_t^{T_n}$ remains a martingale on the set $\{|X_0|\leqslant n\}$). We can also replace the function f with some bounded function $f_{b,n}\in C^2(\mathbb{R})$ which is equal to f on $[-n,n]$. This permits to apply Theorem 4.17, on $\{|X_0|\leqslant n\}$, obtaining

$$\begin{aligned}
f(X_t^{T_n}) &= f_{b,n}(X_t^{T_n})=\\
&= f_{b,n}(X_0)+\int_0^t f'_{b,n}(X_s^{T_n})\mathrm{d}X_s^{T_n}+\frac{1}{2}\int_0^t f''_{b,n}(X_s^{T_n})\mathrm{d}[X^{T_n}]_s\\
&= f(X_0)+\int_0^{t\wedge T_n} f'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^{t\wedge T_n} f''(X_s)\mathrm{d}[X]_s.
\end{aligned}$$

Taking the limit $n\to+\infty$, we get the thesis. □

4.2.2 Multidimensional Ito formula

Ito formula can be generalize to the case of n continuous semimartingales $(X^1,\ldots,X^n)$.

Theorem 4.20. *Let $X^1,\ldots,X^n$ be n semimartingales and let $f\colon\mathbb{R}^n\to\mathbb{R}$ be a $C^2(\mathbb{R}^n,\mathbb{R})$ function then*

$$f(\bar{X}_t)-f(\bar{X}_0)=\sum_{j=1}^n\int_0^t\frac{\partial f}{\partial x^j}(\bar{X}_t)\mathrm{d}X_t^j+\frac{1}{2}\sum_{j,i=1}^n\int_0^t\frac{\partial^2 f}{\partial x^j\partial x^i}(\bar{X}_t)\mathrm{d}[X^i,X^j]_t,$$

where $\bar{X}_t=(X_t^1,\ldots,X_t^n)\in\mathbb{R}^n$. Furthermore if $f\in C^2(\mathbb{R}^n)$ is bounded and $\bar{X}\in\mathcal{M}_c^4$ then $f(\bar{X})$ is in $\mathcal{M}_c^2$.

Proof. The proof is similar to the one of Theorem 4.17 and Theorem 4.19 where Theorem 3.28 equation (3.4) is used instead of Theorem 3.19 equation (3.1). □

4.2.3 Ito processes and Ito formula

Definition 4.21. *Consider $(B^1,\ldots,B^r)$ be r independent Brownian motions, and consider $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ be their completed natural filtration, we say that the process X_t is an Ito process if there is some random variable X_0 whic is $\mathcal{G}_0$ measurable, and there are some progressive processes $\mu_t(\omega),\sigma_t^1(\omega),\ldots,\sigma_t^r(\omega)$ such that, for any $t\geqslant 0$,*

$$\int_0^t|\mu_s|\mathrm{d}s,\int_0^t(\sigma_s^1)^2\mathrm{d}s,\ldots,\int_0^t(\sigma_s^r)^2\mathrm{d}s<+\infty$$

almost surely, for which

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s^1 \mathrm{d}B_s^1 + \cdots + \int_0^t \sigma_s^r \mathrm{d}B_s^r.$$

Theorem 4.22. (Ito formula for Ito processes) *Consider $f \in C^2(\mathbb{R})$ and let X be an Ito process, then $f(X_t)$ is also an Ito process and we have*

$$f(X_t) - f(X_0) = \sum_{k=1}^r \int_0^t f'(X_s)\sigma_s^k \mathrm{d}B_s^k + \int_0^t \left(\mu_s f'(X_s) + \frac{1}{2}\sum_{k=1}^r (\sigma_s^r)^2 f''(X_s)\right)\mathrm{d}s.$$

Proof. The proof follows form the fact that

$$\begin{aligned}[X]_t = [X, X]_t = & \left[\sum_{k=1}^r \int_0^\cdot \sigma_s^k \mathrm{d}B_s^k, \sum_{k'=1}^r \int_0^\cdot \sigma_s^{k'} \mathrm{d}B_s^{k'}\right]_t = \\ = & \sum_{k=1}^r \sum_{k'=1}^r \left[\int_0^\cdot \sigma_s^k \mathrm{d}B_s^k, \int_0^\cdot \sigma_s^{k'} \mathrm{d}B_s^{k'}\right]_t = \\ = & \sum_{k=1}^r \sum_{k'=1}^r \int_0^t \sigma_s^k \sigma_s^{k'} \mathrm{d}[B^k, B^{k'}]_s.\end{aligned}$$

Since

$$[B^k, B^{k'}]_s = \delta_{k,k'} s$$

(the proof is left as an exercise) the theorem is proved. □

More generally if $\sigma_t^{k,j} \colon \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and $\mu^j \colon \mathbb{R}_+ \times \Omega \to \mathbb{R}$ ($k = 1, \ldots, n$ and $j = 1, \ldots, m$) are progressive processsuch that

$$\int_0^t (\sigma_s^{k,j})^2 \mathrm{d}s, \int_0^t |\mu_s^j| \mathrm{d}s < +\infty$$

almost surely for every $k = 1, \ldots, n$ and $j = 1, \ldots, m$ we defined $m \in \mathbb{N}$ Ito processes as

$$X_t^j = X_0^j + \int_0^t \mu_s^j \mathrm{d}s + \sum_{k=1}^n \int_0^t \sigma_s^{k,j} \mathrm{d}B_s^k.$$

Then we have the following theorem.

Theorem 4.23. *Let $\bar{X}_t = (X_t^1, \ldots, X_t^n)$ be an Ito process on $\mathbb{R}^n$ then for any function $f \in C^2(\mathbb{R}^n, \mathbb{R})$ we have*

$$\begin{aligned}f(\bar{X}_t) - f(\bar{X}_0) = & \sum_{k=1}^m \int_0^t \left(\sum_{j=1}^n \frac{\partial f}{\partial x^j}(\bar{X}_s)\sigma_s^{k,j}\right)\mathrm{d}B_s^k + \\ & + \int_0^t \left(\sum_{j=1}^n \frac{\partial f}{\partial x^j}(\bar{X}_s)\mu_s^j + \frac{1}{2}\sum_{j,i=1}^n \sum_{k=1}^m \frac{\partial^2 f}{\partial x^j \partial x^i}(\bar{X}_s)\sigma_s^{k,j}\sigma_s^{k,i}\right)\mathrm{d}s.\end{aligned}$$

Proof. The theorem follows from the fact that

$$\begin{aligned}[X^i, X^j]_t = & \left[\sum_{k=1}^m \int_0^\cdot \sigma_s^{k,i} \mathrm{d}B_s^k, \sum_{k'=1}^m \int_0^\cdot \sigma_s^{k',j} \mathrm{d}B_s^{k'}\right]_t = \\ = & \sum_{k=1}^m \sum_{k'=1}^m \left[\int_0^\cdot \sigma_s^{k,i} \mathrm{d}B_s^k, \int_0^\cdot \sigma_s^{k',j} \mathrm{d}B_s^{k'}\right]_t = \sum_{k=1}^m \sum_{k'=1}^m \int_0^t \sigma_s^{k,i}\sigma_s^{k',j} \mathrm{d}[B_s^k, B_s^{k'}] = \\ = & \sum_{k=1}^m \sum_{k'=1}^m \int_0^t \sigma_s^{k,i}\sigma_s^{k',j}\delta_{k,k'} \mathrm{d}s = \int_0^t \left(\sum_{k=1}^m \sigma^{k,i}\sigma_s^{k,j}\right)\mathrm{d}s.\end{aligned}$$

□

4.3 Other stochastic integrations and their Ito formulas

4.3.1 Backward stochastic integration

Definition 4.24. *Let X_t be a semimartingale and let Z_t be a continuous (adapted) process. We say that Z_t is* backward stochastic integrable *with respect to X_t if for any $\tau > 0$ the following limit exists*

$$\int_0^\tau Z_s \mathrm{d}^- X_s = \lim_{\pi \in \Pi([0,\tau]), |\pi| \to 0} \sum_{t_k \in \pi \setminus \{0\}} Z_{t_k}(X_{t_k} - X_{t_{k-1}})$$

in probability (and it does not depend on the partition $|\pi| \to 0$). When it exists we call $\int_0^\tau Z_s \mathrm{d}^- X_s$ the backward integral *of Z with respect (or driven by) X_t.*

Theorem 4.25. *Let Z and X be two semimartingales, then Z is backward integrable with respect to X and we have*

$$\int_0^\tau Z_s \mathrm{d}^- X_s = \int_0^\tau Z_s \mathrm{d}X_s + [Z, X]_s.$$

Proof. Let $Z = A + N$ and $X = B + M$ (where A, B are continuous bounded variation processes and N, M are local martingales) be the canonical decomposition of Z_t and X_t, respectively, and consider a partition $\pi \in \Pi((0, +\infty))$. Then we have

$$\sum_{t_k \in \pi^t \setminus \{0\}} Z_{t_k}(X_{t_k} - X_{t_{k-1}}) = \sum_{t_k \in \pi^t \setminus \{0\}} Z_{t_k}(B_{t_k} - B_{t_{k-1}}) + \sum_{t_k \in \pi^t \setminus \{0\}} Z_{t_k}(M_{t_k} - M_{t_{k-1}}).$$

Since Z_t is continuous and B_t is of bounded variation and continuous, we have

$$\lim_{|\pi| \to 0} \sum_{t_k \in \pi^t \setminus \{0\}} Z_{t_k}(B_{t_k} - B_{t_{k-1}}) = \int_0^t Z_s \mathrm{d}B_s,$$

where the last integral is in the Riemann-Stieltjes sense. We now want to compute

$$\sum_{t_k \in \pi^t \setminus \{0\}} Z_{t_k}(M_{t_k} - M_{t_{k-1}}) - \sum_{t_k \in \pi^t \setminus \{0\}} Z_{t_k}(M_{t_k} - M_{t_{k-1}}) =$$

$$= \sum_{t_k \in \pi^t \setminus \{0\}} (Z_{t_k} - Z_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) =$$

$$= \sum_{t_k \in \pi^t \setminus \{0\}} (A_{t_k} - A_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) + \sum_{t_k \in \pi^t \setminus \{0\}} (N_{t_k} - N_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}).$$

We have that

$$\sum_{t_k \in \pi^t \setminus \{0\}} (A_{t_k} - A_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) \to 0$$

almost surely. Indeed,

$$\left| \sum_{t_k \in \pi^t \setminus \{0\}} (A_{t_k} - A_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) \right| \leqslant \sum_{t_k \in \pi^t \setminus \{0\}} |A_{t_k} - A_{t_{k-1}}||M_{t_k} - M_{t_{k-1}}|$$

$$\leqslant \left(\sup_{t_k \in \pi \setminus \{0\}} |M_{t_k} - M_{t_{k-1}}| \right) \left(\sum_{t_k \in \pi^t \setminus \{0\}} |A_{t_k} - A_{t_{k-1}}| \right) \leqslant \left(\sup_{t_k \in \pi \setminus \{0\}} |M_{t_k} - M_{t_{k-1}}| \right) V_t(A.).$$

Since $\sup_{t_k \in \pi \setminus \{0\}} |M_{t_k} - M_{t_{k-1}}| \to 0$ almost surely as $|\pi| \to 0$, and $V_t(A) < +\infty$ since A_t has bounded variation, $\sum_{t_k \in \pi^t \setminus \{0\}} (A_{t_k} - A_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) \to 0$.

Since both M, N are both continuous local martingales, we have

$$\sum_{t_k \in \pi^t \setminus \{0\}} (N_{t_k} - N_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) \to [N, M]_t$$

in probability. Since, by definition of quadratic variation of continuous semimartingales, $[Z, X]_t = [N, M]_t$, the theorem is proved. □

Theorem 4.26. (Ito formula for backward integral) *Suppose that X_t is a continuous semimartingale and let $f \in C^3(\mathbb{R})$. Then*

$$f(X_t) - f(X_0) = \int_0^t f'(X_\tau)\mathrm{d}^- X_\tau - \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}[X]_s.$$

Proof. We first prove that the integral $\int_0^t f'(X_\tau)\mathrm{d}^- X_\tau$ is well defined. By Ito formula, since $f \in C^3(\mathbb{R})$ and so $f' \in C^2(\mathbb{R})$, we have that $f'(X_t)$ is a semimartingale such that

$$f'(X_t) = f'(X_0) + \int_0^t f''(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f'''(X_s)\mathrm{d}[X_s].$$

By Theorem 4.25, the integral $\int_0^t f'(X_\tau)\mathrm{d}^- X_\tau$ is well defined and we have

$$\int_0^t f'(X_\tau)\mathrm{d}^- X_\tau = \int_0^t f'(X_s)\mathrm{d}X_s + [f'(X_\cdot), X]_t =$$

$$= \int_0^t f'(X_s)\mathrm{d}X_s + \left[\int_0^\cdot f''(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^\cdot f'''(X_s)\mathrm{d}[X_s], X\right]_t =$$

$$= \int_0^t f'(X_s)\mathrm{d}X_s + \left[\int_0^\cdot f''(X_s)\mathrm{d}X_s, \int_0^\cdot \mathrm{d}X_s\right] = \int_0^t f'(X_s)\mathrm{d}X_s + \int_0^t f''(X_s)\mathrm{d}[X]_s.$$

By Ito formula (for standard Ito integral) we have that

$$f(X_t) - f(X_0) = \int_0^t f'(X_\tau)\mathrm{d}X_\tau + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}[X]_s$$

and so

$$f(X_t) - f(X_0) = \int_0^t f'(X_\tau)\mathrm{d}^- X_\tau - \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}[X]_s.$$ □

4.3.2 Stratonovich and midpoint integral

Definition 4.27. *Let Z_t be a continuous process and X_t be a continuous semimartingale. We say that Z_t is* Stratonovich integrable *with respect to X_t if, for any $\tau > 0$, the following limit*

$$\int_0^\tau Z_s \circ \mathrm{d}X_s = \lim_{\pi \in \Pi([0,\tau]), |\pi| \to 0} \sum_{t_k \in \pi \setminus \{0\}} \left\{\frac{1}{2}(Z_{t_k} + Z_{t_{k-1}})\right\}(X_{t_k} - X_{t_{k-1}})$$

exists in probability, and it does not depend on $\pi \in \Pi([0,\tau])$. When it exists, we call $\int_0^\tau Z_s \circ \mathrm{d}X_s$ the Stratonovich integral *of Z with respect (or driven by) X_t.*

Furthermore, we say that Z_t is mid-point integrable *with respect to X_t if, for any $\tau > 0$, the following limit*

$$\int_0^\tau Z_s \,\tilde{\circ}\, \mathrm{d}X_s = \lim_{\pi \in \Pi([0,\tau]), |\pi| \to 0} \sum_{t_k \in \pi \setminus \{0\}} Z_{\frac{t_k + t_{k-1}}{2}}(X_{t_k} - X_{t_{k-1}})$$

exists in probability, and it does not depend on $\pi \in \Pi([0,\tau])$. When it exists, we call $\int_0^\tau Z_s \,\tilde{\circ}\, \mathrm{d}X_s$ the mid-point integral *of Z with respect (or driven by) X_t.*

It is simple to deduce, from Theorem 4.25 and Theorem 4.26, an existence and Ito formula theorem for Stratonovich integral.

Theorem 4.28. *Let Z_t and X_t be two continuous semimartingale. Then the Stratonovich integral exists and we have*

$$\int_0^t Z_s \circ \mathrm{d}X_s = \int_0^t Z_s \mathrm{d}X_s - \frac{1}{2}[Z, X]_s.$$

Furthermore, if $f \in C^3(\mathbb{R})$ then we have

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) \circ \mathrm{d}X_s.$$

Proof. By Theorem 4.12 (or, better, a generalization of Theorem 4.12 for semimartingales) and Theorem 4.25, we have that

$$\lim_{\pi\in\Pi([0,\tau]),|\pi|\to 0}\sum_{t_k\in\pi\setminus\{0\}}\left\{\frac{1}{2}(Z_{t_k}+Z_{t_{k-1}})\right\}(X_{t_k}-X_{t_{k-1}})=$$

$$=\frac{1}{2}\lim_{\pi\in\Pi([0,\tau]),|\pi|\to 0}\sum_{t_k\in\pi\setminus\{0\}}Z_{t_k}(X_{t_k}-X_{t_{k-1}})+\frac{1}{2}\lim_{\pi\in\Pi([0,\tau]),|\pi|\to 0}\sum_{t_k\in\pi\setminus\{0\}}Z_{t_{k-1}}(X_{t_k}-X_{t_{k-1}})=$$

$$=\frac{1}{2}\int_0^\tau Z_s\mathrm{d}^-X_s+\frac{1}{2}\int_0^\tau Z_s\mathrm{d}X_s=\int_0^\tau Z_s\mathrm{d}X_s+\frac{1}{2}[Z,X].$$

Applying the previous formula in a way similar as the one of the proof of Theorem 4.26, we get that

$$\int_0^t f'(X_\tau)\circ\mathrm{d}X_\tau=\int_0^t f'(X_\tau)\mathrm{d}X_s+\frac{1}{2}\int_0^\tau f''(X_s)\mathrm{d}[X]_s$$

and so

$$f(X_t)-f(X_0)=\int_0^t f'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^\tau f''(X_s)\mathrm{d}[X]_s=\int_0^t f'(X_\tau)\circ\mathrm{d}X_\tau. \qquad \square$$

Remark 4.29. It is important to note that for the Stratonovich integral the fundamental theorem of calculus holds.

Unfortunately for the existence of the mid-point integral it is not enough that the process Z_t is a semimartingale. In what follows, we provide a stronger condition for the existence of the mid-point integral.

Theorem 4.30. *Let X_t and Z_t be two semimartingale such that the process $[X,Z]_t$ is (almost surely) absolutely continuous with respect to the Lebesgue measure. Then Z is mid-point integrable with respect to X and we have*

$$\int_0^\tau Z_s\,\tilde{\circ}\,\mathrm{d}X_s=\int_0^\tau Z_s\circ\mathrm{d}X_s=\int_0^\tau Z_s\mathrm{d}X_s+\frac{1}{2}[X,Z].$$

Remark 4.31. We say that a function $g\colon\mathbb{R}_+\to\mathbb{R}$ is *absolutely continuous*, if there is a function $g'\in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$ such that

$$g(t)-g(0)=\int_0^t g'(s)\mathrm{d}s.$$

The function $g'(t)$ coincides (Lebesgue)-almost everywhere with the derivative of g.

Remark 4.32. The request that the quadratic variation $[Z,X]_t$ is absolutely continuous with respect to Lebesgue is satisfied by any Ito processes. Indeed suppose that X and Z are Ito processes, i.e. there are some progressive processes $\mu_{1,t}(\omega),\sigma^1_{1,t}(\omega),\ldots,\sigma^r_{1,t}(\omega)$ and $\mu_{2,t}(\omega),\sigma^1_{2,t}(\omega),\ldots,\sigma^r_{2,t}(\omega)$ such that, for any $t\geqslant 0$,

$$\int_0^t|\mu_{j,s}|\mathrm{d}s,\int_0^t(\sigma^1_{j,s})^2\mathrm{d}s,\ldots,\int_0^t(\sigma^r_{j,s})^2\mathrm{d}s<+\infty$$

almost surely, for $j\in\{1,2\}$, for which

$$X=X_0+\int_0^t\mu_{1,s}\mathrm{d}s+\sum_{k=1}^r\int_0^t\sigma^k_{1,s}\mathrm{d}B^k_s,$$

$$Z_t=Z_0+\int_0^t\mu_{2,s}\mathrm{d}s+\sum_{k=1}^r\int_0^t\sigma^k_{2,s}\mathrm{d}B^k_s.$$

Then, by the properties of Ito integral with respect to quadratic variation we have that

$$[Z,X]_t=\int_0^t\left(\sum_{k=1}^r\sigma^k_{1,s}\sigma^k_{2,s}\right)\mathrm{d}s.$$

The function

$$\left(\sum_{k=1}^{r} \sigma_{1,s}^{k} \sigma_{2,s}^{k}\right) \in L_{\text{loc}}^{1}(\mathbb{R}_{+}, \mathrm{d}s)$$

almost surely since $\sigma_{j,s}^{k} \in L_{\text{loc}}^{2}(\mathbb{R}_{+}, \mathrm{d}s)$ almost surely.

We now state a generalization of the continuity theorem for $L^1(\mathbb{R}_+, \mathrm{d}t)$ functions. Consider a sequence of functions $\tau_n: \mathbb{R}_+ \to \mathbb{R}$ such that there is $\pi_n \in \Pi((0, +\infty))$ such that $|\pi_n| \to 0$ and a constant $C \in \mathbb{N}$ for which

$$\tau_n(t) = \sum_{t_k^n \in \pi_n \setminus \{0\}} h_{k,n} \mathbb{I}_{[t_{k-1}^n, t_k^n)}(t) \tag{4.8}$$

where

$$|h_{k,n}| \leqslant t_{k+C}^{n} - t_k. \tag{4.9}$$

Lemma 4.33. *Let $a \in L^1(\mathbb{R}_+, \mathrm{d}t)$ and consider a sequence of measurable (bounded) functions τ_n: $\mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions (4.8) and (4.9). Then*

$$\int_{\mathbb{R}_+} |a(t) - a(t + \tau_n(t))| \mathrm{d}t \to 0, \quad n \to +\infty.$$

Proof. Let $a = g$ where $g \in C^0(\mathbb{R}_+, \mathbb{R})$ with compact support contained in $[0, K] \subset \mathbb{R}_+$ (for some $K > 0$). If $\sup_n (\sup_{t \in \mathbb{R}_+} |\tau_n(t)|) = C$ we have that

$$|g(t) - g(t + \tau_n(t))| \leqslant \mathbb{I}_{[0, K+C]}(t) \|g\|_{L^\infty(\mathbb{R})}.$$

Furthermore, by the continuity of g, we have that

$$|g(t) + g(t + \tau_n(t))| \to 0.$$

Thus, by Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}_+} |g(t) - g(t + \tau_n(t))| \mathrm{d}t \to 0, \quad n \to +\infty.$$

Let $a \in L^1(\mathbb{R}_+, \mathrm{d}t)$ be a generic integrable function, then, by the density of continuous functions with compact support in $L^1(\mathbb{R}_+, \mathrm{d}t)$, for any $\varepsilon > 0$ there is $g_\varepsilon \in C^0(\mathbb{R}_+)$ with compact support such that

$$\int_{\mathbb{R}_+} |a(t) - g_\varepsilon(t)| \mathrm{d}t < \varepsilon.$$

Then we have

$$\int_{\mathbb{R}_+} |a(t) - a(t + \tau_n(t))| \mathrm{d}t = \int |a(t) - g(t)| \mathrm{d}t + \int |g(t) - g(t + \tau_n(t))| \mathrm{d}t + \\ + \int_{\mathbb{R}_+} |g(t + \tau_n(t)) - a(t + \tau_n(t))| \mathrm{d}t.$$

Since $\tau_n(t)$ is piecewise constant there is a partition $\pi \in \Pi((0, +\infty))$ such that

$$\tau_n(t) = \sum_{t_k \in \pi \setminus \{0\}} h_k \mathbb{I}_{[t_{k-1}, t_k)}(t)$$

for some $h_k \in \mathbb{R}_+$. Then we have

$$\int_{\mathbb{R}_+} |g(t + \tau_n(t)) - a(t + \tau_n(t))| \mathrm{d}t = \sum_{t_k \in \pi_n \setminus \{0\}} \int_{t_{k-1}}^{t_k} |g(t + h_{k,n}) - a(t + h_{k,n})| \mathrm{d}t = \\ = \sum_{t_k \in \pi_n \setminus \{0\}} \int_{t_{k-1} + h_{k,n}}^{t_k + h_{k,n}} |g(t) - a(t)| \mathrm{d}t \leqslant \sum_{t_k \in \pi_n \setminus \{0\}} \sum_{|t_h - t_{k-1}| \leqslant h_{k,n}} \int_{t_{h-1}}^{t_h} |g(t) - a(t)| \mathrm{d}t.$$

By the hypotheses on τ_n, we have that there is $C \in \mathbb{N}$ such that $h_{k,n} < |t_{k+C} - t_k|$, in other words the sums $\sum_{|t_h - t_{k-1}| \leqslant h_{k,n}} \int_{t_{h-1}}^{t_h} |g(t) - a(t)| \mathrm{d}t$ contains at most C different terms. This means that

$$\int_{\mathbb{R}_+} |g(t + \tau_n(t)) - a(t + \tau_n(t))| \mathrm{d}t \leqslant C \int_{\mathbb{R}_+} |g(t) - a(t)| \mathrm{d}t \leqslant C\varepsilon.$$

This implies that

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}_+}|a(t)-a(t+\tau_n(t))|\mathrm{d}t\right|\leqslant$$

$$\leqslant\limsup_{n\to+\infty}\left|(C+1)\varepsilon+\int_{\mathbb{R}_+}|g(t)-g(t+\tau_n(t))|\mathrm{d}t\right|\leqslant(C+1)\varepsilon.$$

Since $\varepsilon>0$ is arbitrary we proved the theorem. □

Proof. If $Z=A+N$ and $X=B+M$ are the canonical decomposition of Z and X, repeating the proof of Theorem 4.25 we have that,

$$\sum_{t_k\in\pi^t\setminus\{0\}} Z_{\frac{t_k+t_{k-1}}{2}}(X_{t_k}-X_{t_{k-1}})=$$

$$=\sum_{t_k\in\pi^t\setminus\{0\}} Z_{\frac{t_k+t_{k-1}}{2}}(B_{t_k}-B_{t_{k-1}})+\sum_{t_k\in\pi^t\setminus\{0\}} A_{\frac{t_k+t_{k-1}}{2}}(M_{t_k}-M_{t_{k-1}})+$$

$$+\sum_{t_k\in\pi^t\setminus\{0\}} N_{\frac{t_k+t_{k-1}}{2}}(M_{t_k}-M_{t_{k-1}}).$$

Since Z_t is continuous and B is bounded variation we have

$$\lim_{|\pi|\to 0}\sum_{t_k\in\pi^t\setminus\{0\}} Z_{\frac{t_k+t_{k-1}}{2}}(B_{t_k}-B_{t_{k-1}})\to\int_0^t Z_s\mathrm{d}B_s.$$

We have also

$$\left|\sum_{t_k\in\pi^t\setminus\{0\}} A_{\frac{t_k+t_{k-1}}{2}}(M_{t_k}-M_{t_{k-1}})-\sum_{t_k\in\pi^t\setminus\{0\}} A_{t_{k-1}}(M_{t_k}-M_{t_{k-1}})\right|\leqslant$$

$$\leqslant\left\{\sup_{t_k\in\pi^t\setminus\{0\}}|M_{t_k}-M_{t_{k-1}}|\right\}\sum_{t_k\in\pi^t\setminus\{0\}}\left|A_{\frac{t_k+t_{k-1}}{2}}-A_{t_{k-1}}\right|$$

$$\leqslant\left\{\sup_{t_k\in\pi^t\setminus\{0\}}|M_{t_k}-M_{t_{k-1}}|\right\}\sum_{t_k\in\tilde{\pi}^t\setminus\{0\}}|A_{s_k}-A_{s_{k-1}}|\leqslant\left\{\sup_{t_k\in\pi^t\setminus\{0\}}|M_{t_k}-M_{t_{k-1}}|\right\}V_t(A)\to 0$$

almost surely, where $\tilde{\pi}\in\Pi((0,+\infty))$ is defined as $\tilde{\pi}=\pi\cup_{t_k\in\pi\setminus\{0\}}\left\{\frac{t_k+t_{k-1}}{2}\right\}$.

What remains to prove is that, for any increasing sequence $\pi_n\in\Pi((0,+\infty))$ such that $|\pi_n|\to 0$, then

$$\sum_{t_k\in\pi^t\setminus\{0\}} N_{\frac{t_k+t_{k-1}}{2}}(M_{t_k}-M_{t_{k-1}})\to\int_0^t N_s\mathrm{d}M_s+\frac{1}{2}[N,M]_t.$$

First we note that

$$\sum_{t_k\in\pi^t\setminus\{0\}} N_{\frac{t_k+t_{k-1}}{2}}(M_{t_k}-M_{t_{k-1}})=\sum_{t_k\in\pi^t\setminus\{0\}} N_{\frac{t_k+t_{k-1}}{2}}\left\{\left(M_{t_k}-M_{\frac{t_k+t_{k+1}}{2}}\right)+\left(M_{\frac{t_k+t_{k+1}}{2}}-M_{t_{k-1}}\right)\right\}$$

$$=\sum_{t_k\in\pi^t\setminus\{0\}} N_{\frac{t_k+t_{k-1}}{2}}\left(M_{t_k}-M_{\frac{t_k+t_{k+1}}{2}}\right)+\sum_{t_k\in\pi^t\setminus\{0\}} N_{t_{k-1}}\left(M_{\frac{t_k+t_{k+1}}{2}}-M_{t_{k-1}}\right)+$$

$$+\sum_{t_k\in\pi^t\setminus\{0\}}\left(N_{\frac{t_k+t_{k-1}}{2}}-N_{t_{k-1}}\right)\left(M_{\frac{t_k+t_{k+1}}{2}}-M_{t_{k-1}}\right)=$$

$$=\sum_{s_k\in\tilde{\pi}^t\setminus\{0\}} N_{s_{k-1}}(M_{s_k}-M_{s_{k-1}})+\sum_{t_k\in\pi^t\setminus\{0\}}\left(N_{\frac{t_k+t_{k-1}}{2}}-N_{t_{k-1}}\right)\left(M_{\frac{t_k+t_{k+1}}{2}}-M_{t_{k-1}}\right)$$

where $\tilde{\pi}^t=\left(\pi^t\cup_{t_k\in\pi^t\setminus\{0\}}\left\{\frac{t_k+t_{k-1}}{2}\right\}\right)\cap[0,t]$. The sequence

$$\sum_{s_k\in\tilde{\pi}^t\setminus\{0\}} N_{s_{k-1}}(M_{s_k}-M_{s_{k-1}})\to\int_0^t N_s\mathrm{d}M_s$$

in probability. The only thing that we have to prove is that

$$\sum_{t_k\in\pi^t\setminus\{0\}}\left(N_{\frac{t_k+t_{k-1}}{2}}-N_{t_{k-1}}\right)\left(M_{\frac{t_k+t_{k+1}}{2}}-M_{t_{k-1}}\right)\to\frac{1}{2}[N,M]_t$$

in probability. We prove this convergence in the case where $N,M\in\mathcal{M}_c^4$, since the general case can be reduced to this one by localization. First we prove that

$$\sum_{t_k\in\pi^t\setminus\{0\}}\Big(N_{\frac{t_k+t_{k-1}}{2}}-N_{t_{k-1}}\Big)\Big(M_{\frac{t_k+t_{k+1}}{2}}-M_{t_{k-1}}\Big)-\sum_{t_k\in\pi^t\setminus\{0\}}\Big([N,M]_{\frac{t_k+t_{k-1}}{2}}-[N,M]_{t_{k-1}}\Big)$$

converges to 0 in $L^2(\Omega)$. Indeed, repeating the proof of Lemma 4.18 we get

$$\begin{aligned}
\mathbb{E}\Bigg[\Bigg(\sum_{t_k\in\pi^t\setminus\{0\}}\Big(N_{\frac{t_k+t_{k-1}}{2}}-N_{t_{k-1}}\Big)\Big(M_{\frac{t_k+t_{k+1}}{2}}-M_{t_{k-1}}\Big)-\sum_{t_k\in\pi^t\setminus\{0\}}\Big([N,M]_{\frac{t_k+t_{k-1}}{2}}-[N,M]_{t_{k-1}}\Big)\Bigg)^2\Bigg]&\\
\leqslant 3\Bigg(\Bigg[\sup_{s\in[0,t]}(|M_s|+|N_s|)^4\Bigg]^{1/2}+\mathbb{E}[[M]_t^2+[N]_t^2]^{1/2}\Bigg)\times&\\
\times\Bigg(\mathbb{E}\Bigg[\sup_{t_i\in\tilde{\pi}^t\setminus\{0\}}((M_{t_i}-M_{t_{i-1}})^4+(N_{t_{i-1}}-N_{t_i}))\Bigg]^{1/2}+&\\
\mathbb{E}\Bigg[\sup_{t_i\in\tilde{\pi}^t\setminus\{0\}}([M]_{t_i}-[M]_{t_{i-1}}+[N]_{t_i}-[N]_{t_{i-1}})^2\Bigg]^{1/2}\Bigg).&
\end{aligned}$$

which goes to 0 when $|\pi|\to 0$. What remains to prove is that

$$\frac{1}{2}[N,M]_t-\sum_{t_k\in\pi^t\setminus\{0\}}\Big([N,M]_{\frac{t_k+t_{k-1}}{2}}-[N,M]_{t_{k-1}}\Big)\to 0.$$

Since $[N,M]_t$ is absolutely continuous with respect to Lebesgue, there is $a_t(\omega)\in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathrm{d}t)$ such that $[N,M]_t=\int_0^t a_s\mathrm{d}s$. Thus we have

$$\begin{aligned}
&\frac{1}{2}[N,M]_t-\sum_{t_k\in\pi^t\setminus\{0\}}\Big([N,M]_{\frac{t_k+t_{k-1}}{2}}-[N,M]_{t_{k-1}}\Big)=\\
&=\frac{1}{2}\sum_{t_k\in\pi^t\setminus\{0\}}\Big([N,M]_{t_k}-2[N,M]_{\frac{t_k+t_{k-1}}{2}}+[N,M]_{t_{k-1}}\Big)=\\
&=\frac{1}{2}\sum_{t_k\in\pi^t\setminus\{0\}}\Bigg(\int_{\frac{t_k+t_{k-1}}{2}}^{t_k}a_s\mathrm{d}s-\int_{t_{k-1}}^{\frac{t_k+t_{k-1}}{2}}a_s\mathrm{d}s\Bigg)=\\
&=\frac{1}{2}\sum_{t_k\in\pi^t\setminus\{0\}}\Bigg(\int_{t_{k-1}}^{\frac{t_k+t_{k-1}}{2}}a_{s+\frac{t_k-t_{k-1}}{2}}\mathrm{d}s-\int_{t_{k-1}}^{\frac{t_k+t_{k-1}}{2}}a_s\mathrm{d}s\Bigg)=\\
&=\frac{1}{2}\sum_{t_k\in\pi^t\setminus\{0\}}\Bigg(\int_{t_{k-1}}^{\frac{t_k+t_{k-1}}{2}}\Big(a_{s+\frac{t_k-t_{k-1}}{2}}-a_s\Big)\mathrm{d}s\Bigg).
\end{aligned}$$

In other words we have

$$\begin{aligned}
&\left|\frac{1}{2}[N,M]_t-\sum_{t_k\in\pi_n^t\setminus\{0\}}\Big([N,M]_{\frac{t_k+t_{k-1}}{2}}-[N,M]_{t_{k-1}}\Big)\right|\leqslant\\
&\leqslant\frac{1}{2}\int_{\mathbb{R}_+}|\mathbb{I}_{[0,t]}(s+\tau_n(s))a_{t+\tau_n(t)}-\mathbb{I}_{[0,t]}(s)a_s|\mathrm{d}s.
\end{aligned}$$

where the functions $\tau_n(t)$ are defined as

$$\tau_n(t)=\sum_{t_k\in\tilde{\pi},k\quad\text{even}}\frac{t_{k+2}-t_k}{2}\mathbb{I}_{[t_k,t_{k+1})}(t).$$

Obviously τ_n satisfies the conditions (4.8) and (4.9), and thus, by Lemma 4.33 we have

$$\frac{1}{2}\int_{\mathbb{R}_+}|\mathbb{I}_{[0,t]}(s+\tau_n(s))a_{t+\tau_n(t)}-\mathbb{I}_{[0,t]}(s)a_s|\mathrm{d}s.\to 0$$

as $n\to+\infty$, since $\mathbb{I}_{[0,t]}(s)a_s\in L^1(\mathbb{R}_+,\mathrm{d}t)$. This concludes the proof of the theorem. $\square$

Chapter 5

Consequences of Ito formula and Girsanov theorem

5.1 Applications of Ito formula to Brownian motion

5.1.1 Martingale representation theorem

Consider a Brownian motion B_t and let $\mathcal{F}_t^B$ be its natural (in general not completed) filtration. We want to prove that any $L^2(\Omega)$ (cadlag) martingale M_t with respect to the filtration $\mathcal{F}_t^B$ is an Ito process.

We start with the following proposition.

Proposition 5.1. *Let $K \in L^2(\mathcal{F}_t^B)$, then there is a progressive function $h(s) \in L^2(\Omega \times [0,t], \mathrm{d}\mathbb{P}\mathrm{d}t)$ (which is unique up to set of measures 0 with respect the measure $\mathrm{d}\mathbb{P}\mathrm{d}t$) such that*

$$K = \mathbb{E}[K] + \int_0^t h(s)\mathrm{d}B_s. \tag{5.1}$$

First we prove the following lemma.

Lemma 5.2. *Consider the family of functions*

$$\mathcal{J}_t = \mathrm{span}\{e^{i\lambda_1(B_{t_2}-B_{t_1})+\cdots+i\lambda_{n-1}(B_{t_n}-B_{t_{n-1}})} | \lambda_1, \ldots, \lambda_n \in \mathbb{R}, t_1 \leqslant \cdots \leqslant t_n \leqslant t\} \subset L^2(\mathcal{G}_t).$$

Then $\mathcal{J}_t$ is dense in $L^2(\mathcal{F}_t^B)$.

Proof. Since $L^2(\mathcal{F}_t^B)$ is an Hilbert space it is enough to prove that $\mathcal{J}_t^\perp = \{0\}$, namely that if $K \in L^2(\mathcal{F}_t^B)$ such that

$$\mathbb{E}[K e^{i\lambda_1(B_{t_2}-B_{t_1})+\cdots+i\lambda_{n-1}(B_{t_n}-B_{t_{n-1}})}] = 0 \tag{5.2}$$

for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}, t_1 \leqslant \cdots \leqslant t_n \leqslant t$ then $K = 0$ almost surely.

Consider the (bounded) measure $\mu: \mathbb{R}^{n-1} \to \mathbb{R}$ defined as if $F \in \mathcal{B}(\mathbb{R}^{n-1})$ we have

$$\mu(F) = \mathbb{E}[K \mathbb{I}_{\{(B_{t_2}-B_{t_1}, \ldots, B_{t_2}-B_{t_1}) \in F\}}(\omega)].$$

The measure μ is bounded, indeed $|\mu(F)| \leqslant \mathbb{E}[|K|] < +\infty$, and the function

$$\hat{\mu}(\lambda_1, \ldots, \lambda_{n-1}) =$$

$$= \mathbb{E}[K e^{i\lambda_1(B_{t_2}-B_{t_1})+\cdots+i\lambda_{n-1}(B_{t_n}-B_{t_{n-1}})}] = \int_{\mathbb{R}^{n-1}} e^{i\lambda_1 x_1+\cdots+\lambda_{n-1}x_n} \mu(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$$

is its characteristic function (or equivalently its Fourier transform). Condition (5.2) implies that $\hat{\mu}(\lambda_1, \ldots, \lambda_{n-1}) = 0$. Since the characteristic function of bounded measure on $\mathbb{R}^n$ characterizes completely the measure, this implies that $\mu(F) = 0$ for any $F \in \mathcal{B}(\mathbb{R}^{n-1})$. Since $n \in \mathbb{N}$ and $F \in \mathcal{B}(\mathbb{R}^{n-1})$ are generic, and the sets of the form

$$\{(B_{t_2} - B_{t_1}, \ldots, B_{t_2} - B_{t_1}) \in F\} \in \mathcal{F}_t^B$$

generates the σ-algebra $\mathcal{F}_t^B$ (since by definition $\mathcal{F}_t^B = \sigma(B_s | s \leqslant t)$), this means that

$$\mathbb{E}[K|\mathcal{F}_t^B] = 0$$

almost surely. Since K is $\mathcal{F}_t^B$-measurable we have that $K = \mathbb{E}[K|\mathcal{F}_t^B] = 0$ almost surely and then the thesis. $\square$

Lemma 5.3. *Suppose that $K \in \mathcal{J}_t$, then the thesis of Proposition 5.1 holds.*

Proof. Let $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t$ and consider the martingales

$$B_t^j = \int_{t_j \wedge t}^{t_{j+1} \wedge t} \mathrm{d}B_s$$

and the bounded variation processes

$$T^j(t) = \int_{t_j}^{t_{j+1} \wedge t} \mathrm{d}s$$

for $j = 1, \ldots, n-1$. Consider the (complex) processes

$$F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(s), \ldots, T^n(s), B_s^1, \ldots, B_s^n) = \exp\left(\sum_{r=1}^{n-1} \left(i\lambda_r B_s^r + \frac{1}{2}\lambda_r^2 T^r(s)\right)\right).$$

Applying Ito formula to the function F (or better to the real and imaginary part of the function F), we get

$$F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(t), \ldots, T^n(t), B_t^1, \ldots, B_t^n) - 1 =$$

$$= \int_0^t \sum_{r=1}^n \partial_{y^r} F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(t), \ldots, T^n(t), B_t^1, \ldots, B_t^n) \mathrm{d}t +$$

$$+ \int_0^t \sum_{r=1}^n \partial_{y^{r+n}} F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(t), \ldots, T^n(t), B_t^1, \ldots, B_t^n) \mathrm{d}B_s^r +$$

$$+ \frac{1}{2} \int_0^t \sum_{r,r'=1}^n \partial_{y^{r+n}, y^{r'+n}} F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(t), \ldots, T^n(t), B_t^1, \ldots, B_t^n) \mathrm{d}[B^r, B^{r'}]_s. \tag{5.3}$$

By noting that

$$B_s^j = \int_0^s \mathbb{I}_{[t_j, t_{j+1}]}(\tau) \mathrm{d}B_\tau$$

and so

$$[B^j, B^{j'}]_s = \int_0^s \mathbb{I}_{[t_j, t_{j+1}]}(\tau) \mathbb{I}_{[t_{j'}, t_{j'+1}]}(\tau) \mathrm{d}\tau = \delta_{j,j'} \int_0^s \mathbb{I}_{[t_j, t_{j+1}]} \mathrm{d}(\tau) = \delta_{j,j'} T^j(s)$$

and finally that

$$\partial_{y^r} F_{\lambda_1, \ldots, \lambda_{n-1}} = \frac{\lambda_r^2}{2}$$

$$\partial_{y^{r+n}, y^{r'+n}} F_{\lambda_1, \ldots, \lambda_{n-1}} = -\delta_{r,r'} \lambda_r^2,$$

we obtain that the first and last term in the sum (5.3) cancel out, and so

$$F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(t), \ldots, T^n(t), B_t^1, \ldots, B_t^n) - 1 =$$

$$= \int_0^t \sum_{r=1}^n \partial_{y^{r+n}} F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(s), \ldots, T^n(s), B_s^1, \ldots, B_s^n) \mathrm{d}B_s^r =$$

$$= \int_0^t \left(\sum_{r=1}^n \partial_{y^{r+n}} F_{\lambda_1, \ldots, \lambda_{n-1}}(T^1(s), \ldots, T^n(s), B_s^1, \ldots, B_s^n) \mathbb{I}_{[t_r, t_{r+1}]}(s)\right) \mathrm{d}B_s =$$

$$= \int_0^t h(s) \mathrm{d}B_s$$

for the (bounded and continuous) process

$$h(s)=\sum_{r=1}^{n}\partial_{y^{r+n}}F_{\lambda_1,\ldots,\lambda_{n-1}}(T^1(t),\ldots,T^n(t),B^1_t,\ldots,B^n_t)\mathbb{I}_{[t_r,t_{r+1}]}(s).$$

Since

$$F_{\lambda_1,\ldots,\lambda_{n-1}}(T^1(t),\ldots,T^n(t),B^1_t,\ldots,B^n_t)=$$

$$=\exp(i\lambda_1(B_{t_2}-B_{t_1})+\cdots+i\lambda_{n-1}(B_{t_n}-B_{t_{n-1}}))C(t)$$

where $C(t)>0$ is a suitable constant, the lemma is proved. □

Proof. First we prove uniqueness of the representation (5.1). Suppose that there are $h,h'\in L^2_{[0,t]}(B)$ such that

$$K=\mathbb{E}[K]+\int_0^t h(s)\mathrm{d}B_s=\mathbb{E}[K]+\int_0^t h'(s)\mathrm{d}B_s,$$

then we have that

$$0=\mathbb{E}[(K-\mathbb{E}[K]-K+\mathbb{E}[K])^2]=\mathbb{E}\left[\left(\int_0^t(h(s)-h'(s))\mathrm{d}B_s\right)^2\right]=$$

$$=\mathbb{E}\left[\int_0^t(h(s)-h'(s))^2\mathrm{d}s\right],$$

and so $h=h'$ up to a set of measure zero with respect to $\mathrm{d}\mathbb{P}\mathrm{d}t$.

We want to prove the existence. Let $\mathcal{L}\subset L^2(\mathcal{F}^B_t)$ be the following space

$$\mathcal{L}=\left\{k+\int_0^t h(s)\mathrm{d}B_s, k\in\mathbb{R}, h\in L^2_{[0,t]}(B)\right\}.$$

We want to prove that $\mathcal{L}$ is a *closed* subspace of $L^2(\mathcal{F}^B_t)$. Indeed let $P_n\in\mathcal{L}$ be a Cauchy sequence in $L^2(\mathcal{F}^B_t)$, this means that there are some $k_n\in\mathbb{R}$ and $h_n(s)\in L^2_{[0,t]}(B)$ such that

$$P^n=k_n+\int_0^t h_n(s)\mathrm{d}B_s.$$

We have that

$$\mathbb{E}[(P_n-P_m)^2]=\mathbb{E}\left[\left(k_n-k_m+\int_0^t(h_n(s)-h_m(s))\mathrm{d}B_s\right)^2\right]=$$

$$=(k_n-k_m)^2+2(k_n-k_m)\mathbb{E}\left[\int_0^t(h_n(s)-h_m(s))\mathrm{d}B_s\right]+\mathbb{E}\left[\left(\int_0^t(h_n(s)-h_m(s))\mathrm{d}B_s\right)^2\right]=$$

$$=(k_n-k_m)^2+\mathbb{E}\left[\int_0^t(h_n(t)-h_m(t))\mathrm{d}t\right].$$

Since P_n is a Cauchy sequence in $L^2(\mathcal{F}^B_t)$, then also k_n is a Cauchy sequence in $\mathbb{R}$ and h_n is a Cauchy sequence in $L^2_{[0,t]}(B)$, i.e. there is $k\in\mathbb{R}$ and a $h\in L^2_{[0,t]}(\mathbb{R})$ such that $k_n\to k$ in $\mathbb{R}$ and $h_n\to h$ in $L^2_{[0,t]}(B)$. If we write $P=k+\int_0^t h(s)\mathrm{d}B_s$, the previous observations imply that $P_n\to P$ in $L^2(\mathcal{G}_t)$. On the other hand, $P\in\mathcal{L}$ and, since P_n is a generic Cauchy sequence in $\mathcal{L}$, it follows that $\mathcal{L}$ is closed.

Furthermore, by Lemma 5.3, $\mathcal{J}_t\subset\mathcal{L}$, and thus, $\overline{\mathcal{J}_t}\subset\overline{\mathcal{L}}=\overline{\mathcal{L}}$. Finally, by Lemma 5.2, $\overline{\mathcal{J}_t}=L^2(\mathcal{F}^B_t)$ and so $\mathcal{L}=L^2(\mathcal{F}^B_t)$. □

Theorem 5.4. *Let M_t be a $L^2(\Omega)$ martingale with respect to the filtration $\mathcal{F}^B_t$. Then there is a process $h\in L^2(B)$ and $M_0\in\mathbb{R}$ such that*

$$M_t=M_0+\int_0^t h(s)\mathrm{d}B_s$$

(where we identify indistinguishable processes).

Proof. We can consider the sequence of martingales

$$M_t^n = M_{t\wedge n} - M_{t\wedge(n-1)}.$$

We have that $M_n^n \in L^2(\mathcal{G}_n)$ and $M_t - M_0 = \sum_{n=1}^{+\infty} M_t^n$ (the previous sums is always convergent since for any $t>0$ only a finite number of its elements are nonzero). This means that, by Proposition 5.1, there is a sequence $h_n \in L^2_{[0,n]}(B)$ such that

$$M_n^n = \int_0^n h_n(s)\mathrm{d}B_s.$$

Since M_t^n is a martingale we have that, for any $0 \leqslant t \leqslant n$

$$M_t^n = \mathbb{E}[M_n^n|\mathcal{G}_t] = \mathbb{E}\left[\int_0^n h_n(s)\mathrm{d}B_s\middle|\mathcal{F}_t^B\right] = \int_0^t h_n(s)\mathrm{d}B_s,$$

almost surely. Furthermore, since $M_t^n = M_n^n$ for $t \geqslant n$, if we extend $h_n(s) = 0$ when $s > n$, we have that

$$M_t^n = \int_0^n h_n(s)\mathrm{d}B_s = \int_0^t h_n(s)\mathrm{d}B_s.$$

Finally, since $M_t^n = 0$ for $t \leqslant n-1$, by the uniqueness part of Proposition 5.1, we have that

$$h_n(t) = 0$$

for $t < n-1$ and $\mathrm{d}\mathbb{P}\mathrm{d}t$ almost everywhere. So, if we write

$$h(t) = \sum_{n=1}^{+\infty} h_n(t),$$

we have that $h \in L^2(M)$ (since, for any finite $t>0$, only a finite number of terms, in the previous sum, are nonzero) and also

$$M_t - M_0 = \sum_{n=1}^{+\infty} M_t^n = \sum_{n=1}^{+\infty} \int_0^t h_n(s)\mathrm{d}B_s = \int_0^t \sum_{n=1}^{+\infty} h_n(s)\mathrm{d}B_s = \int_0^t h(s)\mathrm{d}B_s.$$

Since M_t is adapted, M_0 must be $\mathcal{F}_0^B$ measurable, and so M_0 is a constant almost surely. In conclusion, for any $t>0$ we have $M_t = M_0 + \int_0^t h(s)\mathrm{d}B_s$ almost surely. This means that the process $M_0 + \int_0^t h(s)\mathrm{d}B_s$ is a modification of M_t. Since $M_0 + \int_0^t h(s)\mathrm{d}B_s$ is continuous the theorem is proved. ☐

Corollary 5.5. *If M_t is a $L^2(\Omega)$ martingale with respect to the filtration $\mathcal{F}_t^B$, then it admits a continuous modification.*

Proof. By the previous theorem, M_t is indistinguishable from $M_0 + \int_0^t h(s)\mathrm{d}B_s$ which is a continuous martingale since is the Ito integral with respect to a continuous martingale. ☐

Corollary 5.6. *Let M_t be a continuous local martingale with respect to $\mathcal{F}_t^B$ then there is $h(s) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{d}t)$ such that*

$$M_t = M_0 + \int_0^t h(s)\mathrm{d}B_s.$$

Proof. By localization, we can reduce to the case where $M^{T_n} \in L^2(\Omega)$, then we apply Theorem 5.4, and then we take $T_n \to +\infty$. ☐

Corollary 5.7. *Let M_t be a continuous local martingale with respect to $\mathcal{F}_t^B$ then it is an Ito process.*

Remark 5.8. It is possible to extend all the previous result replacing $\mathcal{F}_t^B$ with $\mathcal{G}_t^B$ (i.e. the *completed* natural filtration of the Brownian motion B).

Remark 5.9. It is possible to prove (see, e.g., Chapter 5 Section 4 of [10]) a generalization of Corollary 5.6 and Corollary 5.7 in the following sense: suppose that M_t is a local martingale (without supposing the continuity) with respect to the natural filtration $\mathcal{F}_t^B$ (or the completed natural filtration $\mathcal{G}_t^B$) of a Brownian motion B_t, then there is a progressive process $h \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{d}t)$ such that

$$M_t = M_0 + \int_0^t h(s)\mathrm{d}B_s.$$

This means that any local martingale, with respect to the natural filtration $\mathcal{F}_t^B$ (or the completed natural filtration $\mathcal{G}_t^B$) of a Brownian motion, admits a continuous version.

Remark 5.10. It is possible to generalize Proposition 5.1 and Theorem 5.4 to the case where $\mathcal{F}_t^{\bar{B}}$ is natural filtration of a set $\bar{B} = (B_t^1, \ldots, B_t^n)$ of n independent Brownian motions. In this case, the thesis of Proposition 5.1 becomes: there are some processes $h^1(s), \ldots, h^n(s) \in L^2_{[0,t]}(B^1, \ldots, B^n)$ such that

$$K = \mathbb{E}[K] + \sum_{r=1}^n \int_0^t h^r(s)\mathrm{d}B_s^r.$$

The martingale representation theorem takes a similar form.

5.1.2 Lévy characterization of Brownian motion

In this section, we want to prove a charaterization of n-dimensional Brownian motion which will be very useful in the following.

Theorem 5.11. *Let $(M^1, \ldots, M^n)$ be n local martingales such that $(M_0^1, \ldots, M_0^n) = 0$ and*

$$[M^i, M^j]_t = t.$$

Then $(M^1, \ldots, M^n)$ are n independent Brownian motions.

Proof. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and define the functions

$$F_{\lambda_1, \ldots, \lambda_n}(M_t^1, \ldots, M_t^n, t) = \exp\left(i \sum_{k=1}^n \left(i\lambda_k M_t^k + \frac{1}{2}\lambda_k^2 t\right)\right).$$

We have that, for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $F_{\lambda_1, \ldots, \lambda_n}(M_t^1, \ldots, M_t^n, t)$ is a local martingale. Indeed, applying Ito formula, we have

$$F_{\lambda_1, \ldots, \lambda_n}(M_t^1, \ldots, M_t^n, t) - 1 =$$

$$= \int_0^t \sum_{r=1}^n \partial_{y^r}(F_{\lambda_1, \ldots, \lambda_n})(M_s^1, \ldots, M_s^n, s)\mathrm{d}M_s^r + \int_0^t \partial_t F_{\lambda_1, \ldots, \lambda_n}(M_s^1, \ldots, M_s^n, s)\mathrm{d}s +$$
$$+ \frac{1}{2}\int_0^t \sum_{r,r'=1}^n \partial_{y^r y^{r'}}(F_{\lambda_1, \ldots, \lambda_n})(M_s^1, \ldots, M_s^n, s)\mathrm{d}[M^r, M^{r'}]_s =$$

$$= \int_0^t \sum_{r=1}^n \partial_{y^r}(F_{\lambda_1, \ldots, \lambda_n})(M_s^1, \ldots, M_s^n, s)\mathrm{d}M_s^r +$$
$$+ \int_0^t \left(\frac{1}{2}\sum_{r=1}^n \partial_{y^r y^r}(F_{\lambda_1, \ldots, \lambda_n})(M_s^1, \ldots, M_s^n, s) + \partial_t F_{\lambda_1, \ldots, \lambda_n}(M_s^1, \ldots, M_s^n, s)\right)\mathrm{d}s.$$

We have that

$$\partial_{y^r y^r}(F_{\lambda_1, \ldots, \lambda_n}) = -\lambda_r^2 F_{\lambda_1, \ldots, \lambda_n}$$

and

$$\partial_t F_{\lambda_1, \ldots, \lambda_n} = \frac{1}{2}\sum_{r=1}^n \lambda_r^2$$

and thus

$$\frac{1}{2}\sum_{r=1}^{n} \partial_{y^r y^r}(F_{\lambda_1,\ldots,\lambda_n})(M_s^1,\ldots,M_s^n,s)+\partial_t F_{\lambda_1,\ldots,\lambda_n}(M_s^1,\ldots,M_s^n,s)=0.$$

Thus, we have that

$$F_{\lambda_1,\ldots,\lambda_n}(M_t^1,\ldots,M_t^n,t)-1=\int_0^t\sum_{r=1}^{n} \partial_{y^r}(F_{\lambda_1,\ldots,\lambda_n})(M_s^1,\ldots,M_s^n,s)\mathrm{d}M_s^r$$

is a local martingale. Furthermore, since $F_{\lambda_1,\ldots,\lambda_n}(M_t^1,\ldots,M_t^n,t)$ is bounded, it is a real martingale.

We prove that for any $t_1\leqslant\cdots\leqslant t_\ell$ $(M_{t_2}^1-M_{t_1}^1,\ldots,M_{t_2}^n-M_{t_1}^n)$, $\ldots$, $(M_{t_\ell}^1-M_{t_{\ell-1}}^1,\ldots,M_{t_\ell}^n-M_{t_{\ell-1}}^n)$ are independent. Let $A\in\mathcal{F}_{t_{\ell-1}}$ then

$$\exp\left(\frac{1}{2}\sum_{r=1}^{\ell} \lambda_r^2(t_\ell-t_{\ell-1})\right)\mathbb{E}\big[\mathbb{I}_A\exp\big(i\sum \lambda_r(M_{t_\ell}^r-M_{t_{\ell-1}}^r)\big)\big]$$

$$\mathbb{E}\left[\mathbb{I}_A\frac{F_{\lambda_1,\ldots\lambda_n}(M_{t_\ell}^1,\ldots,M_{t_\ell}^n,t_\ell)}{F_{\lambda_1,\ldots\lambda_n}(M_{t_\ell}^1,\ldots,M_{t_\ell}^n,t_\ell)}\right]=$$

$$=\mathbb{E}\left[\frac{\mathbb{I}_A}{F_{\lambda_1,\ldots\lambda_n}(M_{t_\ell}^1,\ldots,M_{t_\ell}^n,t_\ell)}\mathbb{E}[F_{\lambda_1,\ldots\lambda_n}(M_{t_\ell}^1,\ldots,M_{t_\ell}^n,t_\ell)|\mathcal{F}_{t_\ell}]\right]=$$

$$=\mathbb{E}\left[\frac{\mathbb{I}_A}{F_{\lambda_1,\ldots\lambda_n}(M_{t_\ell}^1,\ldots,M_{t_\ell}^n,t_\ell)}F_{\lambda_1,\ldots\lambda_n}(M_{t_{\ell-1}}^1,\ldots,M_{t_{\ell-1}}^n,t_{\ell-1})\right]=\mathbb{E}[\mathbb{I}_A]=\mathbb{P}(A).$$

In other words,

$$\mathbb{E}\big[\mathbb{I}_A\exp\big(i\sum \lambda_r(M_{t_\ell}^r-M_{t_{\ell-1}}^r)\big)\big]=\exp\left(\frac{1}{2}\sum_{r=1}^{\ell} \lambda_r^2(t_\ell-t_{\ell-1})\right)\mathbb{P}(A).$$

Since, by the proof of Lemma 5.2, $\mathrm{span}\{\exp(i\sum\lambda_r(M_{t_\ell}^r-M_{t_{\ell-1}}^r)),\lambda_1,\ldots,\lambda_r\in\mathbb{R}\}$ is dense in $L^2(\sigma((M_{t_\ell}^1-M_{t_{\ell-1}}^1,\ldots,M_{t_\ell}^n-M_{t_{\ell-1}}^1)))$ we have that for any $F\in\sigma((M_{t_\ell}^1-M_{t_{\ell-1}}^1,\ldots,M_{t_\ell}^n-M_{t_{\ell-1}}^1))$ we have

$$\mathbb{P}(A\cap F)=\mathbb{E}[\mathbb{I}_A\,\mathbb{I}_F]=\mathbb{E}[\mathbb{P}(A)\mathbb{I}_F]=\mathbb{P}(A)\mathbb{P}(F).$$

This proves that $(M_{t_\ell}^1-M_{t_{\ell-1}}^1,\ldots,M_{t_\ell}^n-M_{t_{\ell-1}}^1)$ is independent of $\mathcal{F}_{t_{\ell-1}}$, and since M_t^j are adapted, is independent of $(M_{t_2}^1-M_{t_1}^1,\ldots,M_{t_2}^n-M_{t_1}^n)$, $\ldots$, $(M_{t_{\ell-1}}^1-M_{t_{\ell-2}}^1,\ldots,M_{t_{\ell-1}}^n-M_{t_{\ell-2}}^n)$. Repeating the argument for each time we obtain that $(M_{t_2}^1-M_{t_1}^1,\ldots,M_{t_2}^n-M_{t_1}^n)$, $\ldots$, $(M_{t_\ell}^1-M_{t_{\ell-1}}^1,\ldots,M_{t_\ell}^n-M_{t_{\ell-1}}^n)$ are independent. Finally by the previous computation we get

$$\mathbb{E}\big[\exp\big(i\sum \lambda_r(M_{t_\ell}^r-M_{t_{\ell-1}}^r)\big)\big]=\exp\left(\frac{1}{2}\sum_{r=1}^{\ell} \lambda_r^2(t_\ell-t_{\ell-1})\right)$$

and so $(M_{t_\ell}^1-M_{t_{\ell-1}}^1,\ldots,M_{t_\ell}^n-M_{t_{\ell-1}}^n)\sim N(0,(t_\ell-t_{\ell-1})\mathbb{I}_{\mathbb{R}^n})$. Since M_t^j are continuous this prove that M_t are Brownian motions. □

5.2 Girsanov theorem and applications

Remark 5.12. From now on we use the following notation: if A,C are two semimartingales and B is a progressive process we write

$$\mathrm{d}A_t=B_t\mathrm{d}C_t$$

if and only if

$$A_t-A_0=\int_0^t B_s\mathrm{d}C_s.$$

Remark 5.13. With the previous notation if $F:\mathbb{R}^n\to\mathbb{R}$ is a C^2 function and $\bar{X}$ is a semimartingale on $\mathbb{R}^n$ the Ito formula reads

$$\mathrm{d}F(\bar{X})_t=\sum_{k=1}^n \partial_{y^k}F(\bar{X}_t)\mathrm{d}X_t^k+\frac{1}{2}\sum_{k,r=1}^n \partial_{y^k y^r}F(\bar{X}_t)\mathrm{d}[X^k,X^r]_t.$$

Remark 5.14. If A_t, C_t and B_t are as in Remark 5.12, if $\mathrm{d}A_t=B_t\mathrm{d}C_t$ and C_t is a local martingale also A_t is a local martingale.

5.2.1 Preliminaries

Definition 5.15. *A martingale L_t is uniformly integrable if the family of random variables $\{L_t\}_{t\in\mathbb{R}_+}$ is uniformly integrable.*

Theorem 5.16. *Let L_t be a uniformly integrable martingale then the there is a $\mathcal{F}_\infty$-measurable random variable $L_\infty\in L^1(\Omega)$ such that*

$$L_t=\mathbb{E}[L_\infty|\mathcal{F}_t]$$

almost surely.

Proof. See Theorem 3.19 and Theorem 3.21 in [4] (see also Section 4 of [2]). □

Remark 5.17. It is important to note that if L is an uniform integrable *cadlag* martingale we can extend the Doob optional stopping time in the following way: let T be a (generic) stopping time (i.e. we assume that $T=+\infty$ in a set with possibly positive probability) then we have

$$L_T=\mathbb{E}[L_\infty|\mathcal{F}_T].$$

Definition 5.18. *Consider a measure space $(\Omega,\mathcal{F})$ and let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega,\mathcal{F})$, we say that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ (and we write $\mathbb{Q}\ll\mathbb{P}$) if for any $A\in\mathcal{F}$ such that $\mathbb{P}(A)=0$ also $\mathbb{Q}(A)=0$. We say that the measure $\mathbb{Q}$ is equivalent to the measure $\mathbb{P}$ (and we write $\mathbb{Q}\sim\mathbb{P}$) if $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, and $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ (i.e., $\mathbb{Q}\ll\mathbb{P}$ and $\mathbb{P}\ll\mathbb{Q}$).*

Theorem 5.19. (Radon-Nikodym theorem) *Let $(\Omega,\mathcal{F})$ be a measure space and consider two probability measure $\mathbb{P}$ and $\mathbb{Q}$, then $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ if and only if there is $f\in L^1(\Omega,\mathcal{F},\mathbb{P})$ such that*

$$\mathbb{Q}(A)=\int_A f(\omega)\mathrm{d}\mathbb{P}(\omega).$$

Proof. See Chapter 14, Section 14.13 of [9]. □

Remark 5.20. The function f in the thesis of Radon-Nikodym theorem is called *the density (or derivatives) of the measure $\mathbb{Q}$ with respect to* $\mathbb{P}$, and it is unique up to $\mathbb{P}$-zero-measure sets. In the following we use the notation

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}=f.$$

Consider two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ and let $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ be a filtration. We can consider the measures $\mathbb{P}_{\mathcal{F}_t}=\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}_{\mathcal{F}_t}=\mathbb{Q}|_{\mathcal{F}_t}$ (i.e., the probability measures $\mathbb{P}$ and $\mathbb{Q}$ respectively restricted (as set functions) to the σ-algebra $\mathcal{F}_t\subset\mathcal{F}$). If $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, then, for every $t\in\mathbb{R}_+$, $\mathbb{Q}_{\mathcal{F}_t}$ is absolutely continuous with respect to $\mathbb{P}_{\mathcal{F}_t}$ which implies that, for every $t\in\mathbb{R}_+$, there is a $L^1(\Omega,\mathcal{F}_t,\mathbb{P})$ random variable such that

$$\frac{\mathrm{d}\mathbb{Q}_{\mathcal{F}_t}}{\mathrm{d}\mathbb{P}_{\mathcal{F}_t}}=D_t^{\mathbb{Q}}.$$

Here write also

$$\frac{\mathrm{d}\mathbb{Q}_{\mathcal{F}_\infty}}{\mathrm{d}\mathbb{P}_{\mathcal{F}_\infty}} = D_\infty^{\mathbb{Q}}$$

Definition 5.21. *We say that the probability measure $\mathbb{Q}$ is locally absolutely continuous with respect to the measure $\mathbb{P}$ and the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, if, for any $t\in\mathbb{R}_+$, the measure $\mathbb{Q}_{\mathcal{F}_t}$ (i.e. the probability measure $\mathbb{Q}|_{\mathcal{F}_t}$ restricted to the σ-algebra $\mathcal{F}_t$) is absolutely continuous $\mathbb{P}_{\mathcal{F}_t}$.*

If $\mathbb{Q}$ is locally absolutely continuous with respect to $\mathbb{Q}$

Theorem 5.22. *Let $\mathbb{Q}$ be a probability measure locally absolutely continuous with respect to $\mathbb{P}$ and the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, then the process $D_t^{\mathbb{Q}}$ is a martingale with respect to the measure $\mathbb{P}$ and the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$. Furthermore if $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, we have that $D_t^{\mathbb{Q}}$ is uniformly integrable and we have*

$$D_t^{\mathbb{Q}} = \mathbb{E}_{\mathbb{P}}[D_\infty^{\mathbb{Q}}|\mathcal{F}_t] \tag{5.4}$$

$\mathbb{P}$-almost surely.

Proof. Let $t > s \in \mathbb{R}_+$ and consider $A\in\mathcal{F}_s\subset\mathcal{F}_t$, we have

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_A D_t^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\mathbb{I}_A D_t^{\mathbb{Q}}|\mathcal{F}_s]] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_A\mathbb{E}[D_t^{\mathbb{Q}}|\mathcal{F}_s]].$$

On the other hand, since $A\in\mathcal{F}_s$, by definition of density of absolutely continuous measures, we have

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_A D_s^{\mathbb{Q}}],$$

i.e. $\mathbb{E}_{\mathbb{P}}[\mathbb{I}_A\mathbb{E}[D_t^{\mathbb{Q}}|\mathcal{F}_t]] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_A D_s^{\mathbb{Q}}]$. The second part of the theorem can be proved in a similar way taking $t=\infty$, obtaining that

$$\mathbb{E}_{\mathbb{P}}[D_\infty^{\mathbb{Q}}|\mathcal{F}_t] = D_t^{\mathbb{Q}}$$

$\mathbb{P}$-almost surely. By Doob theorem this proves that $D_t^{\mathbb{Q}}$ is an uniformly integrable martingale. □

Proposition 5.23. *Suppose that $D_t^{\mathbb{Q}}$ admits a continuous modification (we denote this modification again by $D_t^{\mathbb{Q}}$ then for any (bounded when $\mathbb{Q}$ is locally absolutely continuous and also unbounded when $\mathbb{Q}$ is absolutely continuous) stopping time T we have*

$$\frac{\mathrm{d}\mathbb{Q}_{\mathcal{F}_T}}{\mathrm{d}\mathbb{P}_{\mathcal{F}_T}} = D_T^{\mathbb{Q}} \tag{5.5}$$

Furthermore if $\mathbb{Q}\sim\mathbb{P}$ we have

$$\inf_{t\in\mathbb{R}_+} D_t^{\mathbb{Q}} > 0$$

$\mathbb{P}$-almost surely.

Proof. We consider the case when $\mathbb{Q}$ is absolutely continuous, the case where $\mathbb{Q}$ is only absolutely continuous can be proved in a similar way. If T is a stopping time, we have that for the (Doob) optional stopping time theorem (extended to possibly infinite stopping time see Remark 5.17) we get

$$\mathbb{E}[D_\infty|\mathcal{F}_T] = \mathbb{E}[\mathbb{E}[D_\infty|\mathcal{F}_t]|\mathcal{F}_T] = \mathbb{E}[D_t|\mathcal{F}_T] = D_T.$$

Equality (5.5) can, then, be proved in a way similar to the one of equality (5.4).

In order to prove the second assertion, consider

$$T_\varepsilon = \inf\{t\geqslant 0, D_t^{\mathbb{Q}}\leqslant\varepsilon\}.$$

The random variable T_ε is a stopping time since it is the first hitting time of an closed for the continuous process $D_t^{\mathbb{Q}}$. By definition of σ-algebra generated by a stopping time we have that, $\{T_\varepsilon < +\infty\}\in\mathcal{F}_{T_\varepsilon}$. This means, for the first part of the theorem,

$$\mathbb{Q}(\{T_\varepsilon < +\infty\}) = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{T_\varepsilon<+\infty\}}D_\infty] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{T_\varepsilon<+\infty\}}\mathbb{E}[D_\infty|\mathcal{F}_{T_\varepsilon}]] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{T_\varepsilon<+\infty\}}D_{T_\varepsilon}]\leqslant\varepsilon,$$

where we used that $D_{T_\varepsilon} \leqslant \varepsilon$ since D_t is cadlag. From the previous inequality follows that

$$\mathbb{Q}\left(\bigcap_{n\in\mathbb{N}} \left\{T_{\frac{1}{n}} < +\infty\right\}\right) = \lim_{n\to+\infty} \mathbb{Q}\left(\left\{T_{\frac{1}{n}} < +\infty\right\}\right) \leqslant \lim_{n\to+\infty} \frac{1}{n} = 0.$$

Since $\mathbb{P}$ is equivalent to $\mathbb{Q}$ we get that

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \left\{T_{\frac{1}{n}} < +\infty\right\}\right) = 0$$

and so $\inf_{t\in\mathbb{R}_+} D_t^{\mathbb{Q}}$ is strictly positive $\mathbb{P}$-almost surely. □

5.2.2 Girsanov theorem in the Brownian motion case

Now we want to consider the case where the filtration $\mathcal{F}_t$ is equal to the (in general not completed) natural filtration of an n-dimensional Brownian motion $(B^1, \ldots, B^n)$ with respect the probability measure $\mathbb{P}$.

Under these assumptions, if $\mathbb{Q}$ is a probability measure locally absolutely continuous with respect to $\mathbb{P}$, the process $D_t^{\mathbb{Q}}$ is a $\mathbb{P}$-martingale and so, by Remark 5.9, there is a continuous version of $D_t^{\mathbb{Q}}$. For this reason from now on we suppose that $D_t^{\mathbb{Q}}$ is a *continuous martingale.*

Lemma 5.24. *Under the assumptions of this section, suppose that D_t is a (strictly) positive local continuous $\mathbb{P}$-martingale, then there is a unique progressive process $\bar{h} = (h_1, \ldots, h_n) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ such that*

$$D_t = \mathcal{E}(\bar{h})_t := D_0 \cdot \exp\left(\sum_{k=1}^n \int_0^t h_k(s) \mathrm{d}B_s^k - \frac{1}{2}\int_0^t \left(\sum_{k=1}^n |h_k(s)|^2\right) \mathrm{d}s\right).$$

Proof. Since D_t is continuous and strictly positive $\frac{1}{D_t} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{d}[D])$ almost surely (since it is bounded almost surely). Then consider

$$L_t = \int_0^t \frac{\mathrm{d}D_s}{D_s}.$$

Since D_t is a continuous local $\mathbb{P}$-martingale also L_t is a continuous local $\mathbb{P}$-martingale. Thus, by Corollary 5.6 and Remark 5.10, there is a (unique) progressive process $\bar{h} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ such that

$$L_t = \sum_{k=1}^n \int_0^t h_k(s) \mathrm{d}B_s^k.$$

Thus we have that

$$\int_0^t \frac{\mathrm{d}[D]_s}{D_s^2} = [L]_t = \int_0^t \left(\sum_{k=1}^n |h_k(s)|^2\right) \mathrm{d}s.$$

Then, by Ito formula applied to $\log(D_t)$, we get

$$\log(D_t) = \log(D_0) + \int_0^t \frac{\mathrm{d}D_s}{D_s} - \frac{1}{2}\int_0^t \frac{\mathrm{d}[D]_s}{D_s^2} = L_t - \frac{1}{2}[L]_t =$$

$$= \sum_{k=1}^n \int_0^t h_k(s) \mathrm{d}B_s^k - \frac{1}{2}\int_0^t \left(\sum_{k=1}^n |h_k(s)|^2\right) \mathrm{d}s.$$

The uniqueness follows from the uniqueness of canonical decomposition of continuous semimartingales and the uniqueness of martingale representation theorem. □

Remark 5.25. It is important to note that if $D_t = \mathcal{E}(\bar{h})_t$ then

$$D_t - D_0 = \sum_{k=1}^n \int_0^t D_s h_k(s) \mathrm{d}B_s^k.$$

Theorem 5.26. *Consider $\mathbb{P}, \mathbb{Q}, D_t^{\mathbb{Q}}$ as above (with the assumption of the beginning of this subsection), suppose also that $D_t^{\mathbb{Q}}$ is almost surely strictly positive, and let $\bar{h}^{\mathbb{Q}} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ be the progressive process such that*

$$D_t^{\mathbb{Q}} = \mathcal{E}(\bar{h}^{\mathbb{Q}})_t,$$

then if M is a continuous local martingale with respect to the measure $\mathbb{P}$ then

$$\tilde{M}_t = M_t - \left[M, \sum_{k=1}^n \int_0^\cdot h_k^{\mathbb{Q}}(s) \mathrm{d}B_s^k\right]_t = M_t - \sum_{k=1}^n \int_0^t h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s$$

is a continuous local martingale with respect to $\mathbb{Q}$.

Lemma 5.27. *Under the hypothesis of Theorem 5.26, if X_t is a continuous stochastic process such that $X_t\, D_t^{\mathbb{Q}}$ is a continuous local martingale under $\mathbb{P}$, then X_t is a continuous local martingale under $\mathbb{Q}$.*

Proof. We first prove that if T is a (bounded) stopping time such that $(X D^{\mathbb{Q}})_t^T = X_t^T D_t^{\mathbb{Q},T}$ is a continuous martingale under $\mathbb{P}$, then X^T is a continuous martingale under $\mathbb{Q}$.

If $X_t^T D_t^{\mathbb{Q},T}$ is a martingale then $X_t^T \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q})$, indeed by definition of martingality $|X_t^T D_t^{\mathbb{Q},T}| \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ and so

$$+\infty > \mathbb{E}_{\mathbb{P}}[|X_t^T D_t^T|] = \mathbb{E}_{\mathbb{P}}[|X_t^T| D_t^T] = \mathbb{E}_{\mathbb{P}}[|X_t^T| \mathbb{E}[D_{t_{\max}}^{\mathbb{Q}} | \mathcal{F}_{T\wedge t}]] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}[|X_t^T| D_{t_{\max}}^{\mathbb{Q}} | \mathcal{F}_{T\wedge t}]] =$$

$$= \mathbb{E}_{\mathbb{P}}[|X_t^T| D_{t_{\max}}^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{Q}}[|X_t^T|]$$

where $t_{\max} > \max(T)$, we used that $X_t^T = X_{T\wedge t}$ is $\mathcal{F}_{T\wedge t}$ measurable, and that, by Proposition 5.23 we have $\mathbb{E}_{\mathbb{P}}[D_{t_{\max}}^{\mathbb{Q}} | \mathcal{F}_{T\wedge t}] = D_{T\wedge t}^{\mathbb{Q}} = D_t^{\mathbb{Q},T}$.

Consider $t > s \in \mathbb{R}_+$, and let $A \in \mathcal{F}_s$. Since $A \cap \{T > s\} = (A^c \cup \{T \leqslant s\}) \in \mathcal{F}_s$, we have

$$\mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge t} D_{T\wedge t}^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_t^T D_t^{\mathbb{Q},T}] =$$

$$= \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_s^T D_s^{\mathbb{Q},T}] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge s} D_{T\wedge s}^{\mathbb{Q}}]$$

Furthermore the set $A \cap \{T > s\} \in \mathcal{F}_{T\wedge s} \subset \mathcal{F}_{T\wedge s}$ and so $\mathbb{I}_{A\cap\{T>s\}}$ is $\mathcal{F}_{T\wedge s}$ (and thus $\mathcal{F}_{T\wedge t}$) measurable. This means that

$$\mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge t} D_{T\wedge t}^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge t} \mathbb{E}_{\mathbb{P}}[D_{t_{\max}}^{\mathbb{Q}} | \mathcal{F}_{T\wedge t}]] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge t} \mathbb{E}_{\mathbb{P}}[D_{t_{\max}}^{\mathbb{Q}} | \mathcal{F}_{T\wedge t}]]$$

$$= \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge t} D_{t_{\max}}^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge t}],$$

and also

$$\mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge s} D_{T\wedge s}^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge s} \mathbb{E}_{\mathbb{P}}[D_{t_{\max}}^{\mathbb{Q}} | \mathcal{F}_{T\wedge s}]] =$$

$$= \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge s} \mathbb{E}_{\mathbb{P}}[D_{t_{\max}}^{\mathbb{Q}} | \mathcal{F}_{T\wedge s}]] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge s} D_{t_{\max}}^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge s}].$$

Thus

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T>s\}} X_t^T] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge t}] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T>s\}} X_{T\wedge s}] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T>s\}} X_s^T]$$

Obviously $\mathbb{I}_{A\cap\{T\leqslant s\}} X_{T\wedge t} = \mathbb{I}_{A\cap\{T\leqslant s\}} X_s = \mathbb{I}_{A\cap\{T\leqslant s\}} X_{T\wedge s}$ and so we get

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T\leqslant s\}} X_t^T] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T\leqslant s\}} X_{T\wedge t}] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T\leqslant s\}} X_{T\wedge s}] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{A\cap\{T\leqslant s\}} X_s^T].$$

Finally we can conclude that

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_A X_t^T] = \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_A X_s^T]$$

for any $A \in \mathcal{F}_s$, which means, since X_s^T is $\mathcal{F}_s$ measurable, that $\mathbb{E}_{\mathbb{Q}}[X_t^T | \mathcal{F}_s] = X_s^T$.

Suppose that $X_t D_t^{\mathbb{Q}}$ is a $\mathbb{P}$-local martingale then there is a sequence of bounded stopping times $T_n \to +\infty$, for which $(X D^{\mathbb{Q}})_t^{T_n}$ is a $\mathbb{P}$-martingale. For the first part of the proof of the lemma X^{T_n} is a $\mathbb{Q}$-martingale. Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, $T_n \to +\infty$ $\mathbb{Q}$-almost surely, which implies that T_n is a localization sequence for X_t (with respect to the measure $\mathbb{Q}$) and so X_t is a $\mathbb{Q}$-local martingale. $\square$

Proof of Theorem 5.26. Suppose that M is a local martingale (with respect to $\mathbb{P}$) and consider $\tilde{M}$. By Ito formula we have that

$$\tilde{M}_t D_t^{\mathbb{Q}} - \tilde{M}_0 D_0^{\mathbb{Q}} = \int_0^t \tilde{M}_s \mathrm{d}D_s^{\mathbb{Q}} + \int_0^t D_s^{\mathbb{Q}} \mathrm{d}\tilde{M}_s + \int_0^t \mathrm{d}[\tilde{M}, D^{\mathbb{Q}}]_s =$$

$$= \int_0^t \tilde{M}_s \mathrm{d}D_s^{\mathbb{Q}} + \int_0^t D_s^{\mathbb{Q}} \mathrm{d}M_s - \sum_{k=1}^n \int_0^t D_s^{\mathbb{Q}} h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s + [M, D^{\mathbb{Q}}]_t =$$

$$= \int_0^t \tilde{M}_s \mathrm{d}D_s^{\mathbb{Q}} + \int_0^t D_s^{\mathbb{Q}} \mathrm{d}M_s - \sum_{k=1}^n \int_0^t D_s^{\mathbb{Q}} h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s + \left[M, \sum_{k=1}^n \int_0^\cdot D_s^{\mathbb{Q}} h_k^{\mathbb{Q}}(s) \mathrm{d}B_s^k \right]_t =$$

$$= \int_0^t \tilde{M}_s \mathrm{d}D_s^{\mathbb{Q}} + \int_0^t D_s^{\mathbb{Q}} \mathrm{d}M_s - \sum_{k=1}^n \int_0^t D_s^{\mathbb{Q}} h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s + \sum_{k=1}^n \int_0^t D_s^{\mathbb{Q}} h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s =$$

$$= \int_0^t \tilde{M}_s \mathrm{d}D_s^{\mathbb{Q}} + \int_0^t D_s^{\mathbb{Q}} \mathrm{d}M_s.$$

Since, by Hypothesis, M_t is a (continuous) $\mathbb{P}$-local martingale and, by Theorem 5.22, $D_s^{\mathbb{Q}}$ is a (continuous) $\mathbb{P}$-martingale we have that $\tilde{M}_t D_t^{\mathbb{Q}}$ is a continuous $\mathbb{P}$-local martingale.

Thus by Lemma 5.27, $\tilde{M}_t$ is a $\mathbb{Q}$-local martingale. □

Remark 5.28. An important consequence of Theorem 5.26 is that, under the assumption of this section, if $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ and X is a $\mathbb{P}$-semimartingale then X is also a $\mathbb{Q}$-semimartingale. Indeed if $X = A + M$ is the canonical decomposition of X (with respect to $\mathbb{Q}$) we have

$$X = A + M = \left(A + \sum_{k=1}^n \int_0^t h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s \right) + \tilde{M}.$$

Since $\tilde{M}$ is a $\mathbb{Q}$-martingale and $(A + \sum_{k=1}^n \int_0^t h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s)$ is a sum of bounded variation processes ($\mathbb{P}$-almost surely and so also $\mathbb{Q}$-almost surely being $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{Q}$) X is a $\mathbb{Q}$-semimartingale with canonical decomposition $(A + \sum_{k=1}^n \int_0^t h_k^{\mathbb{Q}}(s) \mathrm{d}[M, B^k]_s) + \tilde{M}$.

Theorem 5.29. *Under the hypotheses of Theorem 5.26 we have that*

$$\tilde{B}_t^k = B_t^k - \int_0^t h_k(s) \mathrm{d}s$$

is a Brownian motion with respect to the probability $\mathbb{Q}$.

Lemma 5.30. *Under the hypotheses and the notations of Theorem 5.26 we have that the quadratic variation of M (with respect to $\mathbb{P}$) and of $\tilde{M}$ (with respect to $\mathbb{Q}$) coincide.*

Proof. We want to prove that $(\tilde{M}_t^2 - [M]_t) D_t^{\mathbb{Q}}$ is a $\mathbb{P}$-local martingale since, by Lemma 5.27, this implies that $\tilde{M}_t^2 - [M]_t$ is a $\mathbb{Q}$-local martingale. The thesis follows from the definition of quadratic variation for local martingales.

By Ito formula we have that

$$\mathrm{d}((\tilde{M}_t^2 - [M]_t) D_t^{\mathbb{Q}}) = 2\tilde{M}_t D_t^{\mathbb{Q}} \mathrm{d}\tilde{M}_t - D_t^{\mathbb{Q}} \mathrm{d}[M]_t + (\tilde{M}_t^2 - [M]_t) \mathrm{d}D_t^{\mathbb{Q}} + \frac{1}{2} 2 D_t^{\mathbb{Q}} \mathrm{d}[\tilde{M}]_t +$$

$$+ 2\tilde{M}_t \mathrm{d}[\tilde{M}, D^{\mathbb{Q}}] = 2\tilde{M}_t D_t^{\mathbb{Q}} \mathrm{d}M_t - 2\sum_{k=1}^n D_t^{\mathbb{Q}} h_k^{\mathbb{Q}}(t) \mathrm{d}[M, B^k]_t + (\tilde{M}_t^2 - [M]_t) \mathrm{d}D_t^{\mathbb{Q}} +$$

$$+ 2\tilde{M}_t \mathrm{d}\left[M, \sum_{k=1}^n \int_0^\cdot D_s^{\mathbb{Q}} h_k^{\mathbb{Q}}(s) \mathrm{d}B_s^k \right]_t =$$

$$= 2\tilde{M}_t D_t^{\mathbb{Q}} \mathrm{d}M_t - 2\sum_{k=1}^n D_t^{\mathbb{Q}} h_k^{\mathbb{Q}}(t) \mathrm{d}[M, B^k]_t + (\tilde{M}_t^2 - [M]_t) \mathrm{d}D_t^{\mathbb{Q}} + \sum_{k=1}^n 2\tilde{M}_t D_t^{\mathbb{Q}} h_k^{\mathbb{Q}}(t) \mathrm{d}[M, B^k]_t =$$

$$= 2\tilde{M}_t D_t^{\mathbb{Q}} \mathrm{d}M_t + (\tilde{M}_t^2 - [M]_t) \mathrm{d}D_t^{\mathbb{Q}}.$$

This proves that $(\tilde{M}_t^2 - [M]_t)D_t^{\mathbb{Q}}$ is a $\mathbb{P}$-local martingale and so $\tilde{M}_t^2 - [M]_t$ is a $\mathbb{Q}$-local martingale. □

Remark 5.31. Lemma 5.30 easily generalize to the case of *quadratic covariation* of two martingales.

Remark 5.32. By Remark 5.28, Lemma 5.30 and Remark 5.31, we have that if X and Y are two semimartingales with respect to $\mathbb{P}$ (and thus also with respect to $\mathbb{Q}$) the quadratic covariation $[X, Y]$ with respect to $\mathbb{P}$ coincides with the quadratic covariation with respect to $\mathbb{Q}$.

Proof. Since B_t^k is a $\mathbb{P}$-martingale (and so a $\mathbb{P}$-local martingale) and

$$\left[B^k, \sum_{r=1}^n \int_0^{\cdot} h_r(s)\mathrm{d}B_s^r\right]_t = \sum_{r=1}^n \int_0^t h_r(s)\mathrm{d}[B^k, B^r]_s = \sum_{r=1}^n \int_0^t h_r(s)\delta_{k,r}\mathrm{d}s = \int_0^t h_r(s)\mathrm{d}s,$$

by Theorem 5.26, $\tilde{B}_t^k$ are $\mathbb{Q}$-local martingale. Furthermore the quadratic covariation of $\tilde{B}_t^k$ with respect to $\mathbb{Q}$ are equal to the quadratic covariation of B^k with respect to $\mathbb{P}$ and so

$$[\tilde{B}^k, \tilde{B}^r]_t = [B^k, B^r]_t = \delta_{k,r} t.$$

The thesis follows from Levy characterization of Brownian motion. □

5.2.3 The Novikov condition

In general we have not a direct definition of the measure $\mathbb{Q}$ (since it is difficult to describe what a measure on an infinite dimensional space) what it is usually done is *to defined the measure* $\mathbb{Q}$ *through the process* $D_t^{\mathbb{Q}}$ i.e. we defined

$$\frac{\mathrm{d}\mathbb{Q}_{\mathcal{F}_t}}{\mathrm{d}\mathbb{P}_{\mathcal{F}_t}} := D_t^{\mathbb{Q}} = \mathcal{E}(\bar{h})_t = \exp\left(\sum_{k=1}^n \int_0^t h_k(s)\mathrm{d}B_s^k - \frac{1}{2}\int_0^t \left(\sum_{k=1}^n |h_k(s)|^2\right)\mathrm{d}s\right).$$

For a generic $\bar{h}(s) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ ($\mathbb{P}$-almost surely) the process $\mathcal{E}(\bar{h})_t$ is only a local martingale. On the other hand, by Theorem 5.22, the process $\mathcal{E}(\bar{h})_t$ can be the density of some a locally absolutely continuous probability measure $\mathbb{Q}$ only when $\mathcal{E}(\bar{h})_t$ is a continuous martingale (or when $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, $\mathcal{E}(\bar{h})_t$ must be also uniformly integrable). We present here some sufficient criterion for having such a property.

Theorem 5.33. *Suppose that $\bar{h}(s) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ almost surely, then $\mathcal{E}(\bar{h})_t$ is a martingale if for any $t > 0$ we have one of the following conditions hold:*

1. $\mathbb{E}\big[\exp\big(\frac{1}{2}\int_0^t \sum_{k=1}^n (h_k(s))^2 \mathrm{d}s\big)\big] < +\infty$ *(Novikov's criterion);*
2. $\sum_k \int_0^t h_k(s)\mathrm{d}B_s^k$ *is a (real) martingale and* $\mathbb{E}\big[\exp\big(\frac{1}{2}\sum_{k=1}^n \int_0^t h_k(s)\mathrm{d}B_s^k\big)\big] < +\infty$ *(Kazamaki's criterion).*

Furthermore if one of the two previous conditions hold for $t = +\infty$ (when $\bar{h}(s) \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ almost surely) $\mathcal{E}(\bar{h})_t$ is a uniformly integrable martingale.

Lemma 5.34. *Let X be a positive random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider $\mathcal{C} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that there is $\beta \in \mathbb{R}_+$ and, for any Y there is a σ-algebra $\sigma(Y) \subset \mathcal{F}_Y \subset \mathcal{F}$, for which, for any $A \in \mathcal{F}$ we have*

$$\mathbb{E}[|Y|\mathbb{I}_A] \leqslant (\mathbb{E}[\mathbb{I}_A \mathbb{E}[X|\mathcal{F}_Y]])^\beta$$

then the family $\mathcal{C}$ is uniform integrable.

Proof. We recall that if X is a random variable for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ for which for any $F \in \mathcal{F}$ such that $\mathbb{P}(F) < \delta_\varepsilon$, we have $\mathbb{E}[|X|\mathbb{I}_F] < \varepsilon$.

Consider $K > 0$ and let $A = \{|Y| > K\}$ then we have

$$\mathbb{P}(|Y| > K) \leqslant \frac{\mathbb{E}[\mathbb{I}_A|Y|]}{K} \leqslant \frac{(\mathbb{E}[\mathbb{I}_A \mathbb{E}[X|\mathcal{F}_Y]])^\beta}{K} \leqslant \frac{(\mathbb{E}[\mathbb{I}_A X])^\beta}{K} \leqslant \frac{(\mathbb{E}[X])^\beta}{K}$$

Fix $\varepsilon > 0$, then there is K_ε such that $\frac{(\mathbb{E}[X])^\beta}{K} < \delta_{\varepsilon^{1/\beta}}$, and by the previous inequality we get

$$\mathbb{P}(A) = \mathbb{P}(|Y| > K) \leqslant \delta_{\varepsilon^{1/\beta}}.$$

and thus

$$\mathbb{E}[\mathbb{I}_{\{|Y|>K_\varepsilon\}}|Y|] \leqslant (\mathbb{E}[\mathbb{I}_{\{|Y|>K_\varepsilon\}}\mathbb{E}[X|\mathcal{F}_Y]])^\beta \leqslant (\mathbb{E}[\mathbb{I}_{\{|Y|>K_\varepsilon\}}X])^\beta \leqslant (\varepsilon^{1/\beta})^\beta = \varepsilon. \qquad \square$$

Proof. We prove the case with finite t. The case $t = +\infty$ can be proved in a similar way.

We prove that 1.⇒2. and that 2. implies that $\mathcal{E}(\bar{h})$ is a (real) martingale.

If 1 hold then

$$\mathbb{E}\left[\left[\sum_k \int_0^\cdot h_k(s)\mathrm{d}B_s^k\right]_t^2\right] = \mathbb{E}\left[\left(\int_0^t \sum_{k=1}^n (h_k(s))^2 \mathrm{d}s\right)^2\right] \leqslant 8\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \sum_{k=1}^n (h_k(s))^2\mathrm{d}s\right)\right] < +\infty$$

Since the quadratic variation of $\sum_k \int_0^\cdot h_k(s)\mathrm{d}B_s^k$ is in $L^2(\Omega)$, we get that $\sum_k \int_0^\cdot h_k(s)\mathrm{d}B_s^k$ is a $\mathcal{M}_c^4$ martingale, and so a real martingale (and not only a local one). We recall that $\mathcal{E}(\bar{h})_t$ is a positive local martingale and thus (by Fatou lemma) it is a supermartingale. Since $\mathcal{E}(\bar{h})_0 = 1$ by the supermartingale property

$$\mathbb{E}[\mathcal{E}(\bar{h})_t] \leqslant 1.$$

Thus, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{1}{2}\sum_k \int_0^t h_k(s)\mathrm{d}B_s^k\right)\right] &\leqslant (\mathbb{E}[\mathcal{E}(\bar{h})_t])^{1/2}\left(\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \sum_{k=1}^n (h_k(s))^2\mathrm{d}s\right)\right]\right)^{1/2} \\ &\leqslant \left(\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \sum_{k=1}^n (h_k(s))^2\mathrm{d}s\right)\right]\right)^{1/2} < +\infty.\end{aligned}$$

This prove that 1.⇒2.

Suppose that 2. holds. We recall that if $\mathcal{E}(\bar{h})_t$ is a positive local martingale, and so it is a (real) martingale if and only if $\mathbb{E}[\mathcal{E}(\bar{h})_t] = 1$. Writing $L_t = \sum_{k=1}^n \int_0^t h_k(s)\mathrm{d}B_s^k$, L_t is a (real) martingale and since $\exp\left(\frac{1}{2}x\right)$ is a convex function we have $\exp\left(\frac{1}{2}L_t\right)$ is a submartingale and thus, for any stopping time T we get

$$\exp\left(\frac{1}{2}L_{T\wedge t}\right) \leqslant \mathbb{E}\left[\exp\left(\frac{1}{2}L_t\right)\middle|\mathcal{F}_{T\wedge t}\right].$$

Let $0 < a < 1$ and consider a localization sequence T_n^a for the local martingale $\mathcal{E}(a\bar{h})_t$. For any $A \in \mathcal{F}_t$ we have

$$\begin{aligned}\mathbb{E}[\mathbb{I}_A\mathcal{E}(a\bar{h})_{T_n^a\wedge t}] &\leqslant (\mathbb{E}[\mathcal{E}(\bar{h})_{T_n^a\wedge t}])^{a^2}\left(\mathbb{E}\left[\mathbb{I}_A\exp\left(\frac{aL_{T_n^a\wedge t}}{a+1}\right)\right]\right)^{1-a^2} \leqslant \\ &\leqslant \left(\mathbb{E}\left[\mathbb{I}_A\exp\left(\frac{1}{2}L_{T_n^a\wedge t}\right)\right]\right)^{(1-a^2)\frac{2a}{a+1}} \leqslant \left(\mathbb{E}\left[\mathbb{I}_A\mathbb{E}\left[\exp\left(\frac{1}{2}L_t\right)\middle|\mathcal{F}_{T_n^a\wedge t}\right]\right]\right)^{2(1-a)a}\end{aligned}$$

where we use that $\mathbb{E}[\mathcal{E}(\bar{h})_{T_n^a\wedge t}] \leqslant 1$ (since $\mathcal{E}(\bar{h})_t$ is a supermartingale and Jensen inequality applied to the function $x^{\frac{a+1}{2a}}$ (being $\frac{a+1}{2a} > 1$). By Lemma 5.34, this proves that the family of random variables $\{\mathcal{E}(a\bar{h})_{T_n^a\wedge t}\}_{n\in\mathbb{N}}$ is uniformly integrable and so $\mathcal{E}(a\bar{h})_{T_n^a\wedge t} \to \mathcal{E}(a\bar{h})$ in $L^1(\Omega)$. Thus for any $t > s$ we get

$$\mathbb{E}[\mathcal{E}(a\bar{h})_t|\mathcal{F}_s] = \lim_{n\to+\infty}\mathbb{E}[\mathcal{E}(a\bar{h})_{T_n^a\wedge t}|\mathcal{F}_s] = \lim_{n\to+\infty}\mathcal{E}(a\bar{h})_{T_n^a\wedge s} = \mathcal{E}(a\bar{h})_s.$$

Thus for any $a \in (0,1)$, the process $\mathcal{E}(a\bar{h})_t$ is a real martingale.

Finally we get

$$\begin{aligned}1 = \mathbb{E}[\mathcal{E}(a\bar{h})_t] &\leqslant (\mathbb{E}[\mathcal{E}(\bar{h})_t])^{a^2}\left(\mathbb{E}\left[\mathbb{I}_A\exp\left(\frac{aL_{T_n^a\wedge t}}{a+1}\right)\right]\right)^{1-a^2} \leqslant \\ &\leqslant (\mathbb{E}[\mathcal{E}(\bar{h})_t])^{a^2}\left(\mathbb{E}\left[\exp\left(\frac{1}{2}L_t\right)\right]\right)^{2(1-a)a} < +\infty,\end{aligned}$$

and so taking the limit $a\to 0$ we obtain

$$1\leqslant \mathbb{E}[\mathcal{E}(\bar{h})_t]$$

and thus the thesis. □

5.2.4 Some applications

5.2.4.1 Cameron-Martin theorem

The Cameron-Martin theorem can be seen as a special case of Girsanov theorem when the process $h\in L^2(\mathbb{R}_+,\mathbb{R})$ is deterministic.

In this case $\mathcal{E}(h)_t$ is a uniform integrable martingale. Indeed, for any $t\in[0,+\infty]$ we have

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |h(s)|^2\mathrm{d}s\right)\right]\leqslant \exp\left(\frac{1}{2}\int_0^{+\infty}|h(s)|^2\mathrm{d}s\right)<+\infty.$$

So by Novikov condition $\mathcal{E}(h)_t$ is a uniform integrable martingale. This means that the measure $\mathbb{Q}^h$ defined as

$$\frac{\mathrm{d}\mathbb{Q}^h_{\mathcal{F}_t^B}}{\mathrm{d}\mathbb{P}_{\mathcal{F}_t^B}}:=\exp\left(\int_0^t h(s)\mathrm{d}B_s-\frac{1}{2}\int_0^t (h(s))^2\mathrm{d}s\right)$$

is absolutely continuous with respect to $\mathbb{P}$.

Consider a function $F: C^0([0,\tau],\mathbb{R})\to\mathbb{R}$ which is measurable (with respect the Borel σ-algebra of $C^0([0,\tau],\mathbb{R})$) from the space of continuous functions from $[0,\tau]$ into $\mathbb{R}$. We can compose the function F with a Brownian motion B_t obtaining

$$F(B_{[0,\tau]}):=F(B.),$$

(where we denote by $B_{[0,\tau]}$ the restriction of Brownian motion with respect to the time $t\in[0,\tau]$) which is a random variable on defined on the space Ω (since the Brownian motion can be seen as a measurable function from Ω into the space of continuous function $C^0(\mathbb{R}_+,\mathbb{R})$). The random variable $F(B_{[0,\tau]})$ is measurable with respect to $\mathcal{F}_\tau^B=\sigma(B_s|s\leqslant\tau)$ (since we consider simply the restriction of B on the times $t\in[0,\tau]$).

Theorem 5.35. *Let F, h are as above and suppose that $F(B_{[0,\tau]})\in L^1(\Omega)$ then*

$$\mathbb{E}_{\mathbb{P}}[F(B_{[0,\tau]})]=\mathbb{E}_{\mathbb{P}}\left[F\left(\left(B.-\int_0^{\cdot} h(s)\mathrm{d}B_s\right)_{[0,\tau]}\right)\exp\left(\int_0^t h(s)\mathrm{d}B_s-\frac{1}{2}\int_0^t (h(s))^2\mathrm{d}s\right)\right].$$

Proof. By Theorem 5.29 the process $B_t'=B_t-\int_0^t h(s)\mathrm{d}B_s$ is a Brownian motion with respect to the measure $\mathbb{Q}^h$ and so

$$\mathbb{E}_{\mathbb{P}}[F(B_{[0,\tau]})]=\mathbb{E}_{\mathbb{Q}^h}[F(B'_{[0,\tau]})].$$

The theorem follows from the definition of $\mathbb{Q}^h$ and B_t'. □

Thanks to Cameron-Martin theorem above it is possible to derive a integration by parts formula which is quite important in the derivation of Malliavin calculus. We give here a special simple version of it.

Corollary 5.36. *Let $G:\mathbb{R}\to\mathbb{R}$ be a C^1 bounded function with bounded derivative, then, for any $t>0$, we have*

$$\mathbb{E}\left[G(B_t)\left(\int_0^t h(s)\mathrm{d}B_s\right)\right]=\mathbb{E}\left[G'(B_t)\left(\int_0^t h(s)\mathrm{d}s\right)\right] \tag{5.6}$$

Exercise 5.1. Fix $h\in L^2(\mathbb{R}_+,\mathbb{R})$ and $\lambda_0>0$, prove that

$$\exp\left(\lambda_0\left(\sup_{t>0}\left|\int_0^t h(s)\mathrm{d}B_s\right|\right)\right)\in L^1(\Omega).$$

Proof of Corollary 5.36. The function $F_G: C^0([0,t],\mathbb{R})\to\mathbb{R}$ defined as $\gamma\mapsto F(\gamma):=G(\gamma(t))$ is a continuous and bounded function. We can apply Cameron-Martin theorem obtaining

$$\mathbb{E}[G(B_t)]=\mathbb{E}\left[G\left(B_t-\lambda\int_0^t h(s)\mathrm{d}B_s\right)\exp\left(\lambda\int_0^t h(s)\mathrm{d}B_s-\frac{\lambda^2}{2}\int_0^t (h(s))^2\mathrm{d}s\right)\right]. \tag{5.7}$$

Consider $\lambda_0>0$ then, for any $0\leqslant|\lambda|\leqslant\lambda_0$

$$\left|G\left(B_t-\lambda\int_0^t h(s)\mathrm{d}B_s\right)\exp\left(\lambda\int_0^t h(s)\mathrm{d}B_s-\frac{\lambda^2}{2}\int_0^t (h(s))^2\mathrm{d}s\right)\right|\leqslant$$
$$\leqslant\|G\|_{L^\infty}\exp\left(\lambda_0\left(\sup_{t>0}\left|\int_0^t h(s)\mathrm{d}B_s\right|\right)\right)$$

which by Exercise 5.1 is in $L^1(\Omega)$. Furthermore, for any $|\lambda|\leqslant\lambda_0-\varepsilon$ (where $\lambda_0>\varepsilon>0$) we have

$$\left|\partial_\lambda\left(G\left(B_t-\lambda\int_0^t h(s)\mathrm{d}B_s\right)\exp\left(\lambda\int_0^t h(s)\mathrm{d}B_s-\frac{\lambda^2}{2}\int_0^t (h(s))^2\mathrm{d}s\right)\right)\right|\leqslant$$
$$\leqslant\left|G'\left(B_t-\lambda\int_0^t h(s)\mathrm{d}B_s\right)\exp\left(\lambda\int_0^t h(s)\mathrm{d}B_s-\frac{\lambda^2}{2}\int_0^t (h(s))^2\mathrm{d}s\right)\int_0^t h(s)\mathrm{d}B_s\right|+$$
$$+\left|G\left(B_t-\lambda\int_0^t h(s)\mathrm{d}B_s\right)\exp\left(\lambda\int_0^t h(s)\mathrm{d}B_s-\frac{\lambda^2}{2}\int_0^t (h(s))^2\mathrm{d}s\right)\right|\left|\left(\int_0^t h(s)\mathrm{d}B_s-\frac{\lambda^2}{2}\int_0^t (h(s))^2\mathrm{d}s\right)\right|\leqslant$$
$$\leqslant(\|G\|_{L^\infty}+\|G'\|_{L^\infty})\exp\left((\lambda_0-\varepsilon)\left(\sup_{t>0}\left|\int_0^t h(s)\mathrm{d}B_s\right|\right)\right)\left(\sup_{t>0}\left|\int_0^t h(s)\mathrm{d}B_s\right|\right)\leqslant$$
$$\leqslant C_{\lambda_0,\varepsilon}(\|G\|_{L^\infty}+\|G'\|_{L^\infty})\exp\left(\lambda_0\left(\sup_{t>0}\left|\int_0^t h(s)\mathrm{d}B_s\right|\right)\right)$$

where we use that there is a constant $C_{\lambda_0,\varepsilon}$ such that for any $x>0$, $x\exp((\lambda_0-\varepsilon)x)\leqslant C_{\lambda_0,\varepsilon}\exp(\lambda_0 x)$.

Since both $G(B_t-\lambda\int_0^t h(s)\mathrm{d}B_s)\exp\left(\lambda\int_0^t h(s)\mathrm{d}B_s-\frac{\lambda^2}{2}\int_0^t (h(s))^2\mathrm{d}s\right)$ and its derivative with respect to $|\lambda|\leqslant\lambda_0-\varepsilon$ are uniformly bounded by a $L^1(\Omega)$ function, we can exchange the derivative operation with the expectation in expression (5.7). This means that taking the derivative in 0 with respect to λ at both sides of expression (5.7) we get

$$0=-\mathbb{E}\left[G'(B_t)\int_0^t h(s)\mathrm{d}B_s\right]+\mathbb{E}\left[G(B_t)\int_0^t h(s)\mathrm{d}B_s\right].$$ □

Remark 5.37. A particular case of expression (5.6) is when $h(s)=\mathbb{I}_{[0,t]}(s)$ which gives

$$\mathbb{E}[G(B_t)B_t]=t\mathbb{E}[G'(B_t)],$$

which is equivalent to write

$$\int_{\mathbb{R}} G(x)x\gamma_t(\mathrm{d}x)=t\int_{\mathbb{R}} G'(x)\gamma_t(\mathrm{d}x)$$

where $\gamma_t(\mathrm{d}x)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)$.

5.2.4.2 Law of hitting times for Brownian motion with drift

Using the Cameron-Marint theorem is it possible to hitting time of the Brownian motion with a drift $c\in\mathbb{R}$. Let $a>0$ and consider

$$T_{a,c}=\inf\{t\geqslant 0, B_t+ct=a\}. \tag{5.8}$$

Lemma 5.38. (Reflection principle) *Let B_t be a Brownian motion and $a\geqslant 0$ then*

$$\mathbb{P}\left(\left(\sup_{s\leqslant t} B_t\right)\geqslant a\right)=2\mathbb{P}(B_t\geqslant a).$$

Proof. The stopping time $T_a := T_{a,0}$ then the process $X_t = B_{T_a+t} - B_{T_a} = B_{T_a+t} - a$ is a Brownian motion independent of the σ-algebra $\mathcal{F}_{T_a}$ (Proof for exercise). The we have

$$\begin{aligned}
\mathbb{P}\left(\left(\sup_{s\leqslant t} B_t\right) \geqslant a\right) &= \mathbb{P}\left(\left(\sup_{s\leqslant t} B_t\right) \geqslant a, B_t \geqslant a\right) + \mathbb{P}\left(\left(\sup_{s\leqslant t} B_t\right) \geqslant a, B_t < a\right)\\
&= \mathbb{P}(B_t \geqslant a) + \mathbb{P}\left(\left(\sup_{s\leqslant t} B_t\right) \geqslant a, B_t < a\right)\\
&= \mathbb{P}(B_t \geqslant a) + \mathbb{P}\left(\left(\sup_{s\leqslant t} B_t\right) \geqslant a, X_{t-T_a} < 0\right)\\
&= \mathbb{P}(B_t \geqslant a) + \mathbb{P}(T_a \leqslant t, X_{t-T_a} < 0)
\end{aligned}$$

where we have that the equality of sets

$$\{T_a \leqslant t\} = \left\{\left(\sup_{s\leqslant t} B_t\right) \geqslant a\right\}.$$

Since $\{T_a \leqslant t\} \in \mathcal{F}_{T_a}$, X_{t-T_a} is independent of $\mathcal{F}_{T_a}$, and X_{t-T_a} has the same distibution of $-X_{t-T_a}$ (being X a Brownian motion) we get

$$\mathbb{P}(T_a \leqslant t, X_{t-T_a} < 0) = \mathbb{P}(T_a \leqslant t, X_{t-T_a} > 0)$$

and so

$$\mathbb{P}(T_a \leqslant t, X_{t-T_a} < 0) = \mathbb{P}(T_a \leqslant t, B_t - a > 0) = \mathbb{P}(T_a \leqslant t, B_t > a) = \mathbb{P}(B_t > a) = \mathbb{P}(B_t \geqslant a). \qquad \square$$

Corollary 5.39. *We have that the density of the random variable $T_{a,0}$ is*

$$f_{T_{a,0}}(t) = \mathbb{I}_{[0,+\infty)}(t) \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}}.$$

Thanks to Cameron-Martin theorem we can prove that:

Theorem 5.40. *We have that the density of the random variable $T_{a,c}$ is*

$$f_{T_{a,c}}(s) = \mathbb{I}_{[0,+\infty)}(s) \frac{a}{\sqrt{2\pi s^3}} \exp\left(ca - \frac{1}{2}c^2 s - \frac{a^2}{2s}\right).$$

Proof. Consider

$$h_U(t) = c\mathbb{I}_{[0,U]}(t),$$

and the functions

$$F^c_{U,a}(\gamma) = \mathbb{I}_{\{\sup_{s\leqslant U}(\gamma(s)+cs)\geqslant a\}}$$

$$F_{U,a}(\gamma) = \mathbb{I}_{\{\sup_{s\leqslant U}\gamma(s)\geqslant a\}}.$$

It is clear that

$$F_{U,a}\left(\gamma(\cdot) + \int_0^{\cdot} h(s)\mathrm{d}s\right) = F_{U,a}(\gamma(\cdot) + c\,\cdot\,) = F^c_{U,a}(\gamma).$$

Thus we have that

$$\mathbb{P}(T_a \leqslant U) = \mathbb{P}\left(\sup_{s\leqslant U}(B_t + ct) \geqslant a\right) = \mathbb{E}[F^c_{U,a}(B.)] =$$

$$= \mathbb{E}[F_{U,a}(B. + c\,\cdot\,)] = \mathbb{E}\left[F_{U,a}(B.)\exp\left(c\int_0^t \mathrm{d}B_s - \frac{1}{2}c^2 t\right)\right] =$$

$$= \mathbb{E}\left[F_{U,a}(B.)\mathbb{E}\left[\exp\left(c\int_0^t \mathrm{d}B_s - \frac{1}{2}c^2 t\right)\middle|\mathcal{F}_{T_a}\right]\right] = \mathbb{E}\left[F_{U,a}(B.)\exp\left(c\int_0^{T_a} \mathrm{d}B_s - \frac{1}{2}c^2 T_a\right)\right] =$$

$$= \mathbb{E}\left[\mathbb{I}_{\{T_a\leqslant U\}}\exp\left(cB_{T_a} - \frac{1}{2}c^2 T_a\right)\right] = \mathbb{E}\left[\mathbb{I}_{\{T_a\leqslant U\}}\exp\left(ca - \frac{1}{2}c^2 T_a\right)\right]$$

$$= \int_0^U f_{T_{a,0}}(t)\exp\left(ca - \frac{1}{2}c^2 t\right)\mathrm{d}t = \int_0^U \frac{a}{\sqrt{2\pi t^3}}\exp\left(ca - \frac{1}{2}c^2 t - \frac{a^2}{2t}\right)\mathrm{d}t. \qquad \square$$

Chapter 6
Stochastic differential equations

6.1 Definition

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two natural numbers $n, m \in \mathbb{N}$, and consider two (Borel-measurable) maps

$$\mu := (\mu^k)_{k=1,\ldots,m}: \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m,$$

$$\sigma := (\sigma_j^k)_{k=1,\ldots,m, j=1,\ldots,n}: \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^{n\times m} := \mathrm{Mat}(m,n).$$

Let $\bar{B} = (B^1, \ldots, B^n)$ be a n-dimensional Brownian motion and consider the filtration

$$\mathcal{F}_t = \sigma(\mathcal{F}_0, \mathcal{F}_t^{\bar{B}}),$$

where $\mathcal{F}_t^{\bar{B}} = \sigma(B_s | s \leqslant t)$ is the natural σ-algebra generated by the Brownian motion $\bar{B}$ and $\mathcal{F}_0$ is some σ-algebra independent of $\mathcal{F}_t^{\bar{B}}$ for every $t \geqslant 0$ (i.e. $\mathcal{F}_0$ is independent of $\bar{B}$).

Definition 6.1. *Consider $Y = (Y^1, \ldots, Y^m)$ an $\mathcal{F}_0$-measurable random variable taking values in $\mathbb{R}^m$ and let $X := (X^1, \ldots, X^m): \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ be a continuous process adapted with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ (generated by $\mathcal{F}_0$ and $\mathcal{F}_t^{\bar{B}}$). Furthermore, suppose that for the process X_t we have, for any $t \geqslant 0$,*

$$\int_0^t |\mu^k(s, X_s)| \mathrm{d}s, \int_0^t |\sigma_j^k(s, X_s)|^2 \mathrm{d}s < +\infty$$

almost surely. Then we say that X_t is a ***strong solution to the stochastic differential equation with coefficients (μ, σ) driven by the n-dimensional Brownian motion $\bar{B}$ and with initial condition Y*** *if X_t is adapted with respect to the filtration $\sigma(Y, \mathcal{F}_t^{\bar{B}})$ and, for any $t \geqslant 0$ and $k = 1, \ldots, m$, we have*

$$X_t^k = Y^k + \int_0^t \mu^k(s, X_s) \mathrm{d}s + \sum_{j=1}^n \int_0^t \sigma_j^k(s, X_s) \mathrm{d}B_s^j.$$

Remark 6.2. If X_t satisfies Definition 6.1 we, also, say that X_t satisfies the SDE (μ, σ) with initial condition Y. If X_t satisfies the SDE (μ, σ), then X_t is a continuous semimartingale and we write

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t) \cdot \mathrm{d}\bar{B}_t, \quad X_0 = Y.$$

Remark 6.3. Hereafter, we denote by

$$L^0(\mathcal{G})$$

the set of all functions that are measurable with respect to the σ-algebra $\mathcal{G}$.

Remark 6.4. By Definition of strong solution we get that there is a function $F: \mathbb{R}_+ \times L^0(\mathcal{F}_0) \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}$ such that, for any $\tau > 0$, the restriction $F|_{[0,\tau]}: [0, \tau] \times L^0(\mathcal{F}_0) \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}$ is $\mathcal{B}([0,\tau]) \otimes \mathcal{B}(L^0(\mathcal{F}_0)) \otimes \mathcal{F}_\tau^{\bar{B}}$-measurable (i.e. the map F is progressive), and

$$X_t = F(t, Y, B_{[0,t]}).$$

6.1.1 Some examples

6.1.1.1 The geometric Brownian motion

In the case $m = n = 1$, consider the following SDE

$$\mathrm{d}X_t = A X_t \mathrm{d}t + C X_t \mathrm{d}B_t, \quad X_0 = Y \tag{6.1}$$

where $A, B \in \mathbb{R}$. Solutions to equation (6.1) are called geometric Brownian motions. It is possible to find a function $F: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$X_t = F(t, Y, B_t),$$

(note that, in this case, the solution F depends on the Brownian motion B_t only at time t on not on the whole interval $[0,t]$). Indeed, suppose that $F \in C^\infty(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ then, by Ito formula, we have

$$\mathrm{d}X_t = \mathrm{d}F(t, Y, B_t) = \left(\partial_t F(t, Y, B_t) + \frac{1}{2}\partial_B^2 F(t, Y, B_t)\right)\mathrm{d}t + \partial_B F(t, Y, B_t)\mathrm{d}B_t.$$

If we want that $X_t = F(t, Y, B_t)$ is solution to equation (6.1), then we must have

$$Y = F(0, Y, B_0), \tag{6.2}$$

$$AX_t = AF(t, Y, B_t) = \left(\partial_t F(t, Y, B_t) + \frac{1}{2}\partial_B^2 F(t, Y, B_t)\right), \tag{6.3}$$

$$CX_t = CF(t, Y, B_t) = \partial_B F(t, Y, B_t). \tag{6.4}$$

From equation (6.4) we get

$$F(t, Y, B_t) = G(t, Y)e^{CB_t},$$

for some function $G: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. From equation (6.3) we obtain

$$\partial_t G(t, Y) + \frac{1}{2}C^2 G(t, Y) = AG(t, Y),$$

and so

$$G(t, Y) = H(Y)\, e^{\left(A - \frac{1}{2}C^2\right)t}.$$

Finally, by equation (6.2) we get $H(Y) = Y$ and therefore

$$F(t, Y, B_t) = Ye^{\left(A - \frac{1}{2}C^2\right)t + CB_t}.$$

This means that equation (6.1) admits a strong that has the form

$$X_t = Ye^{\left(A - \frac{1}{2}C^2\right)t + CB_t}. \tag{6.5}$$

Remark 6.5. By Theorem 6.16 below, expression (6.5) gives the unique strong solution to equation (6.1).

6.1.1.2 Ornstein–Uhlenbeck process

Let $m = n = 1$ and consider the SDE

$$\mathrm{d}X_t = AX_t + C\mathrm{d}W_t, \quad X_t = Y, \tag{6.6}$$

where $A, C \in \mathbb{R}$. The solutions to equation (6.6) are called Ornstein–Uhlenbeck processes. We can provide an explicit solution to the previous equation. Indeed, consider

$$\tilde{X}_t = e^{-At}X_t,$$

then we have

$$\mathrm{d}\tilde{X}_t = -Ae^{-At}X_t\mathrm{d}t + e^{-At}\mathrm{d}X_t = -Ae^{-At}X_t\mathrm{d}t + Ae^{-At}X_t\mathrm{d}t + Ce^{-At}\mathrm{d}B_t = Ce^{-At}\mathrm{d}B_t.$$

Since $\tilde{X}_0 = e^{-A0}X_0 = Y$, we get that

$$\tilde{X}_t = Y + C\int_0^t e^{-As}\mathrm{d}B_s.$$

In this way, we obtain

$$X_t = e^{At}Y + C\int_0^t e^{A(t-s)}\mathrm{d}B_s. \tag{6.7}$$

From the previous expression it is clear that equation (6.6) has a strong solution which is given by expression (6.7). In this case the function F of Remark 6.4 is given by

$$F(t, Y, B_{[0,t]}) = e^{At}Y + \int_0^t e^{A(t-s)}\mathrm{d}B_s = e^{At}Y + C\lim_{\pi \to 0}\sum_{t_k \in \pi^t \setminus \{0\}} e^{-A(t-t_{k-1})}(B_{t_k} - B_{t_{k-1}}).$$

Differently from the geometric Brownian motion case, the function F giving the solution to Ornstein–Uhlenbeck process equation depends on the values of Brownian motion B on the whole interval $[0,t]$ and not only on the Brownian motion B_t evaluated at the final time t.

6.2 Uniform Lipschitz case

6.2.1 Existence

Definition 6.6. *We say that μ and σ (as in Section 6.1) satisfy a uniform Lipschitz condition (or **assumption A**) if there is a constant K such that, for any $k=1,\ldots,m$, $j=1,\ldots,n$ and any $t\geqslant 0$, we have*

$$|\mu^k(t,x)|\leqslant K(1+|x|),\quad |\sigma_j^k(t,x)|\leqslant K(1+|x|),$$

$$|\mu^k(t,x)-\mu^k(t,y)|\leqslant K|x-y|,\quad |\sigma_j^k(t,x)-\sigma_j^k(t,y)|\leqslant K|y-x|.$$

Remark 6.7. Let $\lambda>0$ we define the space

$$\mathcal{X}_\lambda:=\{X|X \text{ continuous adapted process}, \|X\|_\lambda<+\infty\},$$

where

$$\|X\|_\lambda^2:=\sup_{T\geqslant 0} e^{-2\lambda T}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|X_t|^2\right].$$

Furthermore, if $Y\in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$ we write

$$\mathcal{X}_\lambda:=\{X\in\mathcal{X}_\lambda|X_0=Y\}.$$

Exercise 6.1. Prove that $(\mathcal{X}_\lambda,\|\cdot\|_\lambda)$ is a Banach space (i.e. $\|\cdot\|_\lambda$ is a norm and $\mathcal{X}_\lambda$ is complete with respect to it). Furthermore, show that for any $\lambda>0$ we have

$$\|X\|_\lambda^2\leqslant\mathbb{E}\left[\sup_{t\geqslant 0}\{e^{-2\lambda t}|X_t|^2\}\right]\leqslant\|X\|_{\lambda/2}^2.$$

Theorem 6.8. *Suppose that (μ,σ) satisfy assumption A, and suppose that $Y\in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$. Then there exists $\lambda_0>0$, for which there is a continuous adapted process X_t strong solution to the SDE (μ,σ) with initial condition Y such that, for any $\lambda>\lambda_0$,*

$$\mathbb{E}\left[\sup_{t\geqslant 0}(e^{-2\lambda t}|X_t|^2)\right]<+\infty. \tag{6.8}$$

Furthermore, X_t is the unique strong solution to the SDE (μ,σ) with initial condition Y satisfying (6.8).

Theorem 6.9. (Banach fixed point theorem) *Let $(\mathcal{X},d)$ be a (complete) metric space and let $T\colon\mathcal{X}\to\mathcal{X}$ be a map such that there is $0\leqslant k<1$ for which, for any $x,y\in\mathcal{X}$, we have*

$$d(T(x),T(y))\leqslant k\,d(x,y).$$

Then the map T admits a unique fixed point, i.e. there is only one $\tilde{x}\in\mathcal{X}$ such that

$$T(\tilde{x})=\tilde{x}.$$

Furthermore, for any $x_0\in\mathcal{X}$, we have that the sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{X}$, defined by recursion as

$$x_1=T(x_0),\quad x_{n+1}=T(x_n),$$

converges to $\tilde{x}$.

Proof. The proof can be found in [6] Chapter 9 Theorem 9.23. □

Proof of Theorem 6.8. By Exercise 6.1, for any $\lambda'>0$ and $Y\in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$, the set $\mathcal{X}_{\lambda'}^Y$ with distance $d(\cdot,\cdot)=\|\cdot-\cdot\|_\lambda$ is a complete metric space. Furthermore, if $X\in\mathcal{X}_{\lambda'}^Y$ then

$$\mathbb{E}\left[\sup_{t\geqslant 0}|e^{-2\lambda' t}X_t|^2\right]\leqslant\|X\|_{\lambda'}^2,$$

and thus X satisfies the condition (6.8) for $\lambda=2\lambda'$. This means that if we prove that there is a unique X a strong solution to the SDE (μ,σ) and initial condition Y such that $X\in\mathcal{X}_{\lambda'}^Y$ for any $\lambda'>\lambda_0'$ Theorem 6.8 is proved with $\lambda_0=2\lambda_0'$.

For this reason, we will prove that there is $\lambda_0'>0$ such that for any $\lambda'>\lambda_0'$ the SDE (μ,σ) admits a unique strong solution in $\mathcal{X}_{\lambda'}^Y$.

In order to prove the previous statement we will use Banach fixed point theorem. Consider the map T defined on the set of continuous adapted processes on $\mathbb{R}^m$ and taking values in the set of continuous adapted process on $\mathbb{R}^m$ such that for any continuous adapted process X we have

$$T(X)_\cdot^k = Y + \int_0^\cdot \mu^k(s,X_s)\mathrm{d}s + \sum_{j=1}^n \int_0^\cdot \sigma_j^k(s,X_s)\mathrm{d}B_s^j, \tag{6.9}$$

for $k=1,\ldots,m$. Since μ,σ satisfy assumption A, we have that

$$|\mu^k(s,X_s)|\leqslant K(1+|X_s|), \quad |\sigma_j^k(s,X_s)|^2\leqslant K^2(1+|X_s|)^2,$$

and so the Riemann-Stieltjes integral and the Ito integrals in equation (6.9) are well defined, and thus T is well defined. Furthermore X is a strong solution to the SDE (μ,σ) with initial condition Y if and only if $X=T(X)$ and thus if and only if X is a fix point of the map T. Thus, if T satisfy the hypotheses of Banach fix point theorem on $\mathcal{X}_{\lambda'}^Y$ for λ' big enough, we can exploit Banach fix point theorem for proving that T has a unique fix point in $\mathcal{X}_{\lambda'}^Y$ for λ' big enough, concluding in this way the proof.

In order to apply Theorem 6.9 we have to prove that for any $\lambda'>\lambda_0'$ there is $k<1$ for which for any $X,Z\in\mathcal{X}_{\lambda'}^Y$ we have

1. $T(X)\in\mathcal{X}_{\lambda'}^Y$,
2. $\|T(X)-T(Z)\|_{\lambda'}\leqslant k\|X-Z\|_{\lambda'}$.

Fix $k=1,\ldots,m$, and $\lambda',T>0$, then we have

$$\begin{aligned}
& e^{-2\lambda' T}\mathbb{E}\left[\sup_{t\leqslant T}\left|\sum_{j=1}^n \int_0^t \sigma_j^k(s,X_s)\mathrm{d}B_s^j\right|^2\right] \\
\text{(Doob martingale inequality)} \quad \leqslant\ & e^{-2\lambda' T}\mathbb{E}\left[\left|\sum_{j=1}^n \int_0^T \sigma_j^k(s,X_s)\mathrm{d}B_s^j\right|^2\right] \\
\text{(Ito isometry)} \quad \leqslant\ & e^{-2\lambda' T}\mathbb{E}\left[\sum_{j=1}^n \int_0^T |\sigma_j^k(s,X_s)|^2\mathrm{d}s\right] \\
\text{(Assumption A)} \quad \leqslant\ & e^{-2\lambda' T}\left(\sum_{k=1}^n \mathbb{E}\left[\int_0^T K^2(1+|X_s|)^2\mathrm{d}s\right]\right) \\
\leqslant\ & 2nK^2\int_0^T e^{-2\lambda' T}\mathbb{E}[1+|X_s|^2]\mathrm{d}s \\
\leqslant\ & \frac{nK^2}{\lambda'} + 2nK^2\int_0^T e^{-2\lambda'(T-s)}\left(e^{-2\lambda' s}\mathbb{E}\left[\sup_{\ell\leqslant s}|X_\ell|^2\right]\right)\mathrm{d}s \\
\leqslant\ & \frac{nK^2}{\lambda'} + 2nK^2\left(\int_0^T e^{-2\lambda'(T-s)}\mathrm{d}s\right)\left(\sup_{s\geqslant 0} e^{-2\lambda' s}\mathbb{E}\left[\sup_{\ell\leqslant s}|X_\ell|^2\right]\right) \\
\leqslant\ & \frac{nK^2}{\lambda'} + \frac{nK^2}{\lambda'}(1-e^{-2\lambda' T})\|X\|_{\lambda'}^2 \\
\leqslant\ & \frac{nK^2}{\lambda'}(1+\|X\|_{\lambda'}^2),
\end{aligned}$$

where we used that $\int_0^T e^{-2\lambda'(T-s)}\mathrm{d}s\leqslant\int_0^T e^{-2\lambda' s}\mathrm{d}s$. This implies that

$$\begin{aligned}
\left\|\sum_{k=1}^n \int_0^\cdot \sigma_j^k(s,X_s)\mathrm{d}B_s^j\right\|_\lambda^2 &= \sup_{T\geqslant 0} e^{-2\lambda' T}\mathbb{E}\left[\sup_{t\leqslant T}\left|\sum_{k=1}^n \int_0^t \sigma_j^k(s,X_s)\mathrm{d}B_s^j\right|^2\right] \\
&\leqslant \frac{nK^2}{\lambda'}(1+\|X\|_{\lambda'}^2).
\end{aligned}$$

We have also that, for any $\lambda' \geqslant 1$,

$$\begin{aligned}
& e^{-2\lambda' T}\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t \mu^k(s,X_s)\mathrm{d}s\right|^2\right]\\
\leqslant\ & e^{-2\lambda' T}\mathbb{E}\left[\sup_{t\leqslant T}\left(\int_0^t |\mu^k(s,X_s)|\mathrm{d}s\right)^2\right]\\
\leqslant\ & e^{-2\lambda' T}\mathbb{E}\left[\left(\int_0^T |\mu^k(s,X_s)|\mathrm{d}s\right)^2\right]\\
\leqslant\ & e^{-2\lambda' T}\mathbb{E}\left[\left(\int_0^T \frac{(1+|T-s|)}{(1+|T-s|)}|\mu^k(s,X_s)|\mathrm{d}s\right)^2\right]\\
\text{(Cauchy-Schwarz ineq.)}\quad \leqslant\ & \left(\int_0^T \frac{1}{1+|T-s|^2}\mathrm{d}s\right)e^{-2\lambda' T}\mathbb{E}\left[\int_0^T (1+|T-s|)^2|\mu^k(s,X_s)|^2\mathrm{d}s\right]\\
\leqslant\ & 4K^2\,\mathrm{artan}(T)\int_0^T e^{-2\lambda'(T-s)}(1+|T-s|^2)\{e^{-2\lambda' s}(1+\mathbb{E}[|X_s|^2])\}\mathrm{d}s\\
\leqslant\ & 2K^2\pi\left(\int_0^T e^{-2\lambda'(T-s)}(1+|T-s|^2)\mathrm{d}s\right)(1+\|X\|_{\lambda'}^2)\\
\leqslant\ & 2K^2\pi\left(\int_0^{+\infty} e^{-2\lambda' s}(1+s^2)\mathrm{d}s\right)(1+\|X\|_{\lambda'}^2)\\
\leqslant\ & \frac{CK^2\pi}{\lambda'}(1+\|X\|_{\lambda'}^2),
\end{aligned}$$

where we use that

$$\int_0^{+\infty} e^{-2\lambda' s}(1+s^2)\mathrm{d}s \leqslant \frac{1}{2\lambda'}\int_0^{+\infty} e^{-s}\left(1+\frac{s^2}{4\lambda'^2}\right)\mathrm{d}s \leqslant \frac{1}{2\lambda'}\int_0^{+\infty} e^{-s}(1+s^2)\mathrm{d}s \leqslant \frac{C}{2\lambda'},$$

for some constant $C>0$ (not depending on $\lambda'\geqslant 1$). This means that, for $\lambda'\geqslant 1$,

$$\begin{aligned}
\|T(X)\|_{\lambda'} \leqslant\ & \sum_{k=1}^m \|T(X)^k\|_{\lambda'}\\
\leqslant\ & m\|Y\|_{L^2(\Omega)} + \sum_{k=1}^m \left\|\int_0^{\cdot}\mu^k(s,X_s)\mathrm{d}s\right\|_{\lambda'} + \sum_{k=1}^m \left\|\sum_{j=1}^n \int_0^{\cdot}\sigma_j^k(s,X_s)\mathrm{d}B_s^j\right\|_{\lambda'}\\
\leqslant\ & m\left(\|Y\|_{L^2(\Omega)} + \left[\sqrt{\frac{nK^2}{\lambda'}} + \sqrt{\frac{CK^2\pi}{\lambda'}}\right](1+\|X\|_{\lambda'})\right),
\end{aligned}$$

which is bounded when $X\in\mathcal{X}_{\lambda'}^Y$. This prove that $T(X)\in\mathcal{X}_{\lambda'}^Y$ when $X\in\mathcal{X}_{\lambda'}^Y$ for $\lambda'\geqslant 1$.

Suppose now that $X,Z\in\mathcal{X}_{\lambda'}^Y$, fix $k=1,\ldots,m$ and $\lambda'\geqslant 1$, then, using the same steps as above and the second condition of assumption A, we get

$$\begin{aligned}
& e^{-2\lambda' T}\mathbb{E}\left[\sup_{t\leqslant T}\left|\sum_{j=1}^n \int_0^t \sigma_j^k(s,X_s)\mathrm{d}B_s^j - \sum_{k=1}^n \int_0^t \sigma_j^k(s,Z_s)\mathrm{d}B_s^j\right|^2\right]\\
\leqslant\ & e^{-2\lambda' T}\mathbb{E}\left[\sum_{j=1}^n \int_0^T (\sigma_j^k(s,X_s)-\sigma(s,Z_s))^2\mathrm{d}s\right]\\
\leqslant\ & \sum_{j=1}^n K^2\int_0^T e^{-2\lambda' T}\mathbb{E}[|X_s-Z_s|^2]\mathrm{d}s\\
\leqslant\ & \frac{nK^2}{2\lambda'}\|X-Z\|_{\mathcal{X}_{\lambda'}^Y}.
\end{aligned}$$

Furthermore, we have, for $\lambda' \geqslant 1$,

$$e^{-2\lambda' T}\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t \mu^k(s,X_s)\mathrm{d}s - \sum_{k=1}^n \int_0^t \mu^k(s,Z_s)\mathrm{d}s\right|^2\right] \leqslant \frac{CK^2\pi}{2\lambda'}\|X\|_{\lambda'}^2.$$

Using the previous inequality, we get

$$\begin{aligned}\|T(X)-T(Z)\|_{\lambda'} &\leqslant \sum_{k=1}^m \left\|\int_0^\cdot \mu^k(s,X_s)\mathrm{d}s - \int_0^\cdot \mu^k(s,Z_s)\mathrm{d}s\right\|_{\lambda'} \\ &\quad + \sum_{k=1}^m \left\|\sum_{j=1}^n \int_0^\cdot \sigma_j^k(s,X_s)\mathrm{d}B_s^j - \sum_{j=1}^n \int_0^\cdot \sigma_j^k(s,Z_s)\mathrm{d}B_s^j\right\|_{\lambda'} \\ &\leqslant \left(\frac{m\sqrt{n\,K^2}+m\sqrt{\pi^2\,K^2C}}{\sqrt{2\lambda'}}\right)\|X-Z\|_{\lambda'}.\end{aligned}$$

This means that if $\lambda' > \max(1, m^2 n\,K^2 + m^2\pi^2\,K^2C)$ we have that there is $k<1$ for which

$$\|T(X)-T(Z)\|_{\lambda'} \leqslant k\|X-Z\|_{\lambda'}. \qquad \square$$

6.2.2 Uniqueness

It is possible to improve the uniqueness result proved in Theorem 6.8.

First, we introduce the notion of weak solution of a SDE.

Definition 6.10. *Let $\bar{B}_t$ be a ($\mathbb{R}^n$ dimensional-)Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ and let $\{\mathcal{J}_t\}_{t\in\mathbb{R}_+}$ be a filtration. We say that $\bar{B}_t$ is a $\mathcal{J}_t$-Brownian motion if*

1. *$\bar{B}_t$ is adapted with respect to $\{\mathcal{J}_t\}_{t\in\mathbb{R}_+}$;*
2. *for every $\tau>0$, the sigma algebra $\sigma(\bar{B}_t - \bar{B}_\tau | t\geqslant\tau)$ is independent of $\mathcal{J}_t$.*

Remark 6.11. If $\mathcal{J}_t = \mathcal{F}_t = \sigma(\mathcal{F}_0, \mathcal{F}_t^{\bar{B}})$ (i.e. the filtration introduced in Section 6.1), then the n-dimensional Brownian motion $\bar{B}$ is also a $\mathcal{J}_t = \mathcal{F}_t$ Brownian motion.

Definition 6.12. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a filtration $\{\mathcal{J}_t\}_{t\in\mathbb{R}_+}$. We say that a pair $(X,\bar{B})$ of ($\mathcal{J}_t$)-adapted continuous processes is **a weak solution to the SDE with coefficient (μ,σ) with initial condition Y** if $\bar{B}$ is a n dimensional $\mathcal{J}_t$-Brownian motion and if, for any $t\geqslant 0$ and $k=1,\ldots,m$, we have*

$$X_t^k = Y^k + \int_0^t \mu^k(s,X_s)\mathrm{d}s + \sum_{j=1}^n \int_0^t \sigma_j^k(s,X_s)\mathrm{d}B_s^j. \tag{6.10}$$

We can introduce a notion of uniqueness for weak solution.

Definition 6.13. *Let (μ,σ) and Y as in Section 6.1 and $(\Omega,\mathcal{F},\mathbb{P})$ and $\{\mathcal{J}_t\}_{t\in\mathbb{R}_+}$ as in Definition 6.12, we say that the SDE (μ,σ) with initial condition Y satisfies the pathwise uniqueness if, for any n dimensional $\mathcal{J}_t$-Brownian motion $\bar{B}$, if $(X,\bar{B})$ and $(X',\bar{B})$ are two weak solutions to the SDE (μ,σ) with initial condition Y we have that X and $\bar{X}$ are indistinguishable.*

Remark 6.14. It is important to note thatr Definition 6.13 must hold for any probability space $(\Omega,\mathcal{F},\mathbb{P})$. In other words if (μ,σ) is an SDE satisfying pathwise uniqueness for any probability space $(\Omega,\mathcal{F},\mathbb{P})$ and filtration $\{\mathcal{J}_t\}_{t\in\mathbb{R}_+}$, and weak solutions $(X,\bar{B})$ and $(X',\bar{B})$ such that $X_0 = X_0'$ then X and X' are indistinguishable.

Remark 6.15. Suppose that a SDE (μ,σ) with initial condition Y satisfies pathwise uniqueness and it admits a strong solution, then the strong solution is unique (if we identify processes if they are equal up to set $\Omega_1\subset\Omega$ of measure 0). This also implies that any weak solution $(X,\bar{B})$ is indistinguishable by the (unique) strong solution X driven by the Brownian motion $\bar{B}$.

Theorem 6.16. *Suppose that (μ,σ) satisfy assumption A, Y be a $\mathcal{J}_0$-measurable random variable, then the SDE (μ,σ) with initial condition Y satisfies pathwise uniqueness.*

First we recall the (integral) Gronwall lemma.

Lemma 6.17. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a (locally bounded) measurable function from $\mathbb{R}_+$ into $\mathbb{R}_+$ and $\alpha,\beta\in\mathbb{R}_+$ such that*

$$f(t)\leqslant\alpha+\int_0^t\beta f(s)\mathrm{d}s,$$

then we have

$$f(t)\leqslant\alpha\exp(\beta t)$$

Proof. Exercise. □

Proof of Theorem 6.16. Let $(X,\bar{B})$ and $(X',\bar{B})$ two weak solutions to the SDE (μ,σ) with initial condition Y. Consider the continuous adapted process $Z_t=X_t-X_t'$ and, for any $\ell\in\mathbb{N}_0$, the stopping time

$$T_\ell=\inf\{t,|Z_t|\geqslant\ell\}.$$

Then we have that the process $|Z_t^{T_\ell}|\leqslant\ell$ (since $Z_0=0$) and it satisfies the differential relation

$$\mathrm{d}Z_t^{T_\ell}=(\mu(t,X_t^{T_\ell})-\mu(t,X'^{T_\ell}_t))\mathrm{d}t+(\sigma(t,X_t^{T_\ell})-\sigma(t,X'^{T_\ell}_t))\cdot\mathrm{d}\bar{B}_t.$$

By Ito formula, using the fact that $Z_0=0$, that $\int_0^\tau(\sigma(t,X_t^{T_\ell})-\sigma(t,X'^{T_\ell}_t))\cdot\mathrm{d}\bar{B}_t$ is an $L^2(\Omega)$ martingale and that $Z_t^{T_\ell}$ is bounded, we get

$$\mathbb{E}[|Z_t^{T_\ell}|^2]=2\sum_{k=1}^m\mathbb{E}\left[\int_0^t(\mu^k(t,X_s^{T_\ell})-\mu^k(t,X'^{T_\ell}_s))Z_s^{k,T_\ell}\mathrm{d}s+\sum_{j=1}^n(\sigma_j^k(s,X_t^{T_\ell})-\sigma_j^k(s,X'^{T_\ell}_s))^2\mathrm{d}s\right].$$

Using assumption A and Young inequality we obtain

$$\mathbb{E}[|Z_t^{T_\ell}|^2]\leqslant 2m(K+nK^2)\int_0^t\mathbb{E}[|Z_s^{T_\ell}|^2]\,\mathrm{d}s$$

If we denote by $f_\ell(t):=\mathbb{E}[|Z_t^{T_\ell}|^2]$ the previous inequality is equivalent to write

$$f_\ell(t)\leqslant 2m(K+nK^2)\int_0^t f_\ell(s)\,\mathrm{d}s,$$

which, by Gronwall lemma for $\alpha=0$ and $\beta=2m(K+nK^2)$, implies that $f_\ell(t)=0$. This means that, for every $t\geqslant 0$, $Z_t^{T_\ell}=0$ almost surely, and, since $Z_t^{T_\ell}$ is continuous, that the process Z^{T_ℓ} is indistinguishable from 0. Since $T_\ell>0$ almost surely, this implies that $T_\ell=+\infty$ almost surely and that $Z=X-X'$ is indistinguishable from 0. □

6.3 Weak solutions and Girsanov theorem

6.3.1 Tanaka counterexample

We have introduced strong solutions introducing a distinction between them and weak solutions (i.e. solutions which are not necessarily adapted to the driving Brownian motion).

We now propose an example which does not admit strong solutions. Consider $n=m=1$ and let X be a one-dimensional Brownian motion and consider

$$B_t=\int_0^t\mathrm{sign}(X_s)\mathrm{d}X_s,$$

where $\operatorname{sign}(x)=1$ if $x\geqslant 0$ and $\operatorname{sign}(x)=1$ otherwise. By Lèvy's characterization of Brownian motion the process $(B_t)_t$ is again a Brownian motion. Moreover, we have

$$\int_0^t \operatorname{sign}(X_s)\mathrm{d}B_s=\int_0^t \operatorname{sign}(X_s)^2\mathrm{d}X_s=\int_0^t \mathrm{d}X_s=X_t-X_0,$$

and so X satisfies the SDE

$$\mathrm{d}X_t=\operatorname{sign}(X_t)\mathrm{d}B_t, \tag{6.11}$$

with B as driving noise and $\sigma(x)=\operatorname{sign}(x)$. This coefficient is not Lipschitz (not even continuous…), and thus we cannot apply Theorem 6.8. Furthermore, if we consider the initial condition $X_0=0$, then we have that both $(X_t)_{t\geqslant 0}$ and $(-X_t)_{t\geqslant 0}$ are solutions, i.e. path-wise uniqueness do not hold for (6.11).

However, solutions of the SDE (6.11) cannot be arbitrary. Indeed, let (X,B) be any solution to equation (6.11), then we have that X_t is a martingale and also

$$[X,X]_t=\int_0^t (\operatorname{sign}(X_s))^2\mathrm{d}s=t.$$

This means that, by Levy characterization of Brownian motion, the process X_t-X_0 must be a Brownian motion. We call this kind of uniqueness, uniqueness in law. More formally:

Definition 6.18. *Let (μ,σ) and Y as in Section 6.1, we say that* ***the SDE (μ,σ) with respect to the initial condition Y satisfies the uniqueness in law property*** *if for any weak solutions $(X,\bar{B})$ and $(X',\bar{B}')$ such that $X_0\sim X_0'\sim Y$ then $X\sim X'$ (i.e. the processes X and X' has the same law).*

Remark 6.19. If X and X' are two (continuous) processes we say that they have the same law if, for any (Borel)-measurable bounded function $F\colon C^0(\mathbb{R}_+,\mathbb{R}^m)\to\mathbb{R}$, we have

$$\mathbb{E}[F(X)]=\mathbb{E}[F(X')].$$

This is also equivalent to say that for any Borel measurable set $A\in\mathcal{B}(C^0(\mathbb{R}_+,\mathbb{R}^m))$ (i.e. $C^0(\mathbb{R}_+,\mathbb{R}^m)$ is equipped with the topology of uniform convergence on compact sets) we have

$$\mathbb{P}(X_\cdot\in A)=\mathbb{E}[\mathbb{I}_A(X)]=\mathbb{E}[\mathbb{I}_A(X')]=\mathbb{P}(X'_\cdot\in A).$$

We want now to prove that if (X,B) is a weak solution of the SDE (6.11) with initial condition $X_0=0$, then the process X_t cannot be adapted with respect to the filtration $\{\mathcal{F}_t^B\}_{t\in\mathbb{R}_+}$ generated by B (and thus X_t cannot be a strong solution to equation (6.11)).

Recall that if φ is a smooth function, by Itô formula we have

$$\varphi(X_t)=\varphi(X_0)+\int_0^t \varphi'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^t \varphi''(X_s)\mathrm{d}s.$$

Take now $\varphi=\varphi_\varepsilon$ even with $\varphi_\varepsilon(x)=(\varepsilon+x^2)^{1/2}$. Then $\varphi_\varepsilon'(x)=x(\varepsilon+x^2)^{-1/2}$ and

$$\varphi_\varepsilon''(x)=(\varepsilon+x^2)^{-1/2}-x^2(\varepsilon+x^2)^{-3/2}=\frac{\varepsilon}{(\varepsilon+x^2)^{3/2}}.$$

Suppose that X_t is a solution to equation (6.11) and consider the process $Z_t^\varepsilon=\int_0^t\varphi_\varepsilon'(X_s)\mathrm{d}X_s$ we have, for any $\tau>0$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant\tau}|Z_t^\varepsilon-B_t|^2\right] &= \mathbb{E}|Z_\tau^\varepsilon-B_\tau|^2=\mathbb{E}\left[\int_0^\tau|\varphi_\varepsilon'(X_s)-\operatorname{sign}(X_s)|^2\mathrm{d}s\right]\\ &= \int_0^\tau\mathbb{E}[|\varphi_\varepsilon'(X_s)-\operatorname{sign}(X_s)|^2]\mathrm{d}s\to 0,\end{aligned}$$

as $\varepsilon \to 0$ by dominated convergence, since $\varphi_\varepsilon'(x) = x(\varepsilon + x^2)^{-1/2} \to \mathrm{sign}(x)$ if $x \neq 0$ and is uniformly bounded so the pointwise (in s) convergence $\mathbb{E}|\varphi_\varepsilon'(X_s) - \mathrm{sign}(X_s)| \to \mathbb{E}[\mathbb{1}_{X_s=0}] = \mathbb{P}(X_s = 0)$, since X_s has the law of a Brownian motion, allows to conclude.

As a consequence, there is a subsequence $\varepsilon_n \to 0$ by subsequences $Z_t^{\varepsilon_n} \to B_t$ uniformly almost surely in any bounded interval. Since φ_ε is even, we also have

$$Z_t^\varepsilon = \varphi_\varepsilon(X_t) - \frac{1}{2}\int_0^t \varphi_\varepsilon''(X_s)\mathrm{d}s = \varphi_\varepsilon(|X_t|) - \frac{1}{2}\int_0^t \varphi_\varepsilon''(|X_s|)\mathrm{d}s.$$

Therefore $(Z_t^\varepsilon)_{t\geqslant 0}$ is actually a function of $|X_t|$. We conclude that $(B_t)_{t\geqslant 0}$ is measurable (and adapted) with respect to the filtration $\mathcal{F}_t^{|X|} = \sigma(|X_s|, s \leqslant t)$. In particular, this proves that $(X_t)_{t\geqslant 0}$ cannot be a strong solution to the SDE (6.11) since otherwise we will have the following inclusion of completed filtrations

$$(\mathcal{F}_t^X)_{t\geqslant 0} \subseteq (\mathcal{F}_t^B)_{t\geqslant 0} \subseteq \left(\mathcal{F}_t^{|X|}\right)_{t\geqslant 0}$$

which is absurd since knowing the modulus of a Brownian motion does not allow to recover its sign. We must conclude that X is strictly a weak solution. And that this holds for all weak solutions. So no strong solutions exists.

What we have discussed is Tanaka's example of a weak solution of an SDE with bounded coefficients which is however not strong. This shows that some regularity of the coefficients is needed to ensure existence of strong solutions.

6.3.2 Building weak solutions with Girsanov theorem

We return now to the concept of uniqueness in law introduced in Definition 6.18.

Let $(X_t)_{t\geqslant 0}$ be a n-dimensional Brownian motion starting at $X_0 = y \in \mathbb{R}^m$ and $b: \mathbb{R}^m \to \mathbb{R}^m$ a measurable vector field growing at most linearly at infinity: i.e. there is a constant K such that

$$|b(x)| \leqslant K(1 + |x|).$$

Then the process

$$Z_t = \exp\left(\int_0^t b(X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^t |b(X_s)|^2 \mathrm{d}s\right), \quad t \geqslant 0, \tag{6.12}$$

is a positive local martingale (and therefore a supermartingale).

Lemma 6.20. (Extended Novikov's condition) *Consider a (progressive random) process $\theta_\cdot \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ almost surely and let X_t be a m-dimensional Brownian motion. Suppose that there is a partition $\pi \subset \Pi([0, +\infty))$ such that, for any $t_k \in \pi \setminus \{0\}$, we have*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{k-1}}^{t_k} |\theta_s|^2 \mathrm{d}s\right)\right] < +\infty, \tag{6.13}$$

then the process

$$\mathcal{E}_t^X(\theta) = \exp\left(\int_0^t \theta_s \mathrm{d}X_s - \frac{1}{2}\int_0^t |\theta_s|^2 \mathrm{d}s\right)$$

is a real martingale (and not only a local martingale).

Proof. Since $\mathcal{E}_t^X(\theta)$ is a positive local martingale, and thus a positive supermartingale, proving the lemma is equivalent to deduce that, for any $t \geqslant 0$, we have

$$\mathbb{E}[\mathcal{E}_t^X(\theta)] = 1.$$

Let $t \leqslant t_k$ then we have that

$$\mathcal{E}_t^X(\theta) = \prod_{r=1}^k \mathcal{E}_t(\theta_\cdot \mathbb{I}_{[t_{r-1} t_r)}(\cdot)).$$

By Novikov's condition we have that $\mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))$ are real martingales, furthermore $\mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))$ is $\mathcal{F}_{t_r}$ measurable. Thus we get

$$
\begin{aligned}
\mathbb{E}[\mathcal{E}_t^X(\theta)] &= \mathbb{E}\left[\prod_{r=1}^{k} \mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{r=1}^{k} \mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))\middle|\mathcal{F}_{t_{k-1}}\right]\right]\\
&= \mathbb{E}\left[\prod_{r=1}^{k-1} \mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))\mathbb{E}[\mathcal{E}_t(\theta.\mathbb{I}_{[t_{k-1}t_k)}(\cdot))|\mathcal{F}_{t_{k-1}}]\right]\\
&= \mathbb{E}\left[\prod_{r=1}^{k-1} \mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))\mathcal{E}_{t_{k-1}}(\theta.\mathbb{I}_{[t_{k-1}t_k)}(\cdot))\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{r=1}^{k-1} \mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))\middle|\mathcal{F}_{t_{k-2}}\right]\right]\\
&= \mathbb{E}\left[\prod_{r=1}^{k-2} \mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))\mathbb{E}[\mathcal{E}_t(\theta.\mathbb{I}_{[t_{k-2},t_{k-1})}(\cdot))|\mathcal{F}_{t_{k-2}}]\right]\\
&= \mathbb{E}\left[\prod_{r=1}^{k-2} \mathcal{E}_t(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot))\right]\\
&\;\vdots\\
&= \mathbb{E}[\mathcal{E}_t(\theta.\mathbb{I}_{[0,t_1)}(\cdot))] = 1,
\end{aligned}
$$

where we used the fact that for any $r \geqslant 1$ we have $\mathcal{E}_{t_{r-1}}(\theta.\mathbb{I}_{[t_{r-1}t_r)}(\cdot)) = 1$. □

Lemma 6.21. *Let X_t be a m dimensional Brownian motion, starting at $X_0 = y \in \mathbb{R}^m$, and let b: $\mathbb{R}^m \to \mathbb{R}^m$ be a measurable map growing at most linearly at infinity. Then Z_t defined in (6.12) is a real (not only local) martingale.*

Proof. We want to apply Lemma 6.20 to the process $\theta_t = b(X_t)$.

Fix $t \geqslant 0$ and $\delta > 0$, then we have

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_t^{t+\delta} |b(X_r)|^2 \mathrm{d}r\right)\right] &\leqslant \mathbb{E}\left[\exp\left(\frac{K^2}{2}\int_t^{t+\delta} (1+|X_r|)^2 \mathrm{d}r\right)\right]\\
&\leqslant \mathbb{E}\left[\exp\left(\frac{3K^2}{2}\int_t^{t+\delta} (1+x+|X_r-x|^2) \mathrm{d}r\right)\right]\\
&\leqslant \mathbb{E}\left[\exp\left(\frac{3K^2\delta(1+x)}{2}\right)\exp\left(\frac{3K^2}{2}\int_t^{t+\delta} \delta|X_r-x|^2 \frac{\mathrm{d}r}{\delta}\right)\right]\\
&\leqslant \mathbb{E}\left[\exp\left(\frac{3K^2\delta(1+x)}{2}\right)\frac{1}{\delta}\int_t^{t+\delta} \exp\left(\frac{3K^2}{2}\delta|X_r-x|^2\right)\mathrm{d}r\right]\\
&\leqslant \exp\left(\frac{3K^2\delta(1+x)}{2}\right)\mathbb{E}\left[\exp\left(\frac{3K^2}{2}\delta|X_{t+\delta}-x|^2\right)\right],
\end{aligned}
$$

which is finite whenever $3\delta K^2 < \frac{1}{t+\delta}$. When $t = 0$ we can take $\delta < \sqrt{\frac{1}{3K^2}}$ and when $t > \frac{1}{2}\sqrt{\frac{1}{3K^2}}$ we can take

$$
\delta \leqslant \left(\frac{1}{3K^2 t}\right)\frac{1}{1+\sqrt{1+\frac{8}{\sqrt{3}K}}} < \frac{-3K^2 t + \sqrt{9K^4t^2 + 12K^2}}{6K^2}.
$$

We consider now the sequence

$$
t_0 = 0, t_1 = \frac{2}{3}\sqrt{\frac{1}{3K^2}}, t_2 = t_1 + \left(\frac{1}{3K^2 t_1}\right)\frac{1}{1+\sqrt{1+\frac{8}{\sqrt{3}K}}}, \ldots, t_k = t_{k-1} + \left(\frac{1}{3K^2 t_{k-1}}\right)\frac{1}{1+\sqrt{1+\frac{8}{\sqrt{3}K}}}.
$$

We have that the seuquence $t_k \to +\infty$ as $k \to +\infty$. Indeed, by inequality (6.21), we have that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_k}^{t_{k+1}}|b(X_r)|^2\mathrm{d}r\right)\right] = \exp\left(\frac{3K^2\delta_k(1+x)}{2}\right)\mathbb{E}\left[\exp\left(\frac{3K^2}{2}\delta_k|X_{t_k+\delta_k}-x|^2\right)\right] < +\infty$$

where $\delta_0 = \frac{2}{3}\sqrt{\frac{1}{3K^2}}$ and $\delta_k = \left(\frac{1}{3K^2 t_k}\right)\frac{1}{1+\sqrt{1+\frac{8}{\sqrt{3}K}}}$. Thus the hypotheses of Lemma 6.20 hold for $\mathcal{E}^X(b(X_\cdot))$. Since $Z_t = \mathcal{E}^X(b(X_\cdot))_t$ the lemma is proved. □

We can then consider the measure $\mathbb{Q}$ defined as, for any $\tau \geqslant 0$,

$$\mathbb{Q}(A) = \mathbb{E}[Z_\tau \mathbb{1}_A], \qquad A \in \mathcal{F}_\tau.$$

Since by Lemma 6.21 the process Z_t is a martingale, $\mathbb{Q}$ is a new measure on Ω which is locally absolutely continuous with respect to $\mathbb{P}$.

Theorem 6.22. *Let X be a m-dimensional Brownian motion with respect to the probability $\mathbb{P}$ starting at $y \in \mathbb{R}^m$, let $b: \mathbb{R}^m \to \mathbb{R}^m$ be a measurable function with at most linear growth at infinity, and consider the process*

$$B_t = X_t - x - \int_0^t b(X_s)\mathrm{d}s. \tag{6.14}$$

Then the pair of processes (X, B) is a weak solution of the SDE $(\mu, \sigma) = (b, I_{m\times m})$ with (deterministic) initial condition $y \in \mathbb{R}^m$ with respect the probability $\mathbb{Q}$ defined in equation (6.21).

Proof. By Girsanov theorem B_t is a Brownian motion with respect to the probability $\mathbb{Q}$. Furthermore if the drift of the SDE $\mu(x) = b(x)$ and the diffusion matrix $\sigma(x) = I_{m\times m}$ (the identity matrix in $\mathrm{Mat}(m,m)$) equation (6.14) is equivalent to the fact that (X, B) satisfies equation

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\cdot \mathrm{d}B_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t.$$ □

6.3.3 About uniqueness in law

Theorem 6.23. *Suppose that the SDE (μ, σ) with initial condition Y satisfies pathwise uniqueness property then it satisfies the uniqueness in law property.*

Proof. (See also Chapter IX Theorem 1.7 of [5]) Suppose that $(X, \bar{B})$ and $(X', \bar{B}')$ are two weak solution of the SDE (μ, σ) we want to build a new probability space Ω' containing both the solution X and X' driven by only one Brownian motion $\bar{B}$.

Consider

$$\Omega' = C^0(\mathbb{R}_+, \mathbb{R}^m) \times C^0(\mathbb{R}_+, \mathbb{R}^m) \times C^0(\mathbb{R}_+, \mathbb{R}^n),$$

with the Borel σ-algebra $\mathcal{F}$. On Ω' we defined a probability law $\mathbb{P}'$, induced by the probability $\mathbb{P}$, of the form

$$\mathbb{P}'(\mathrm{d}\omega_1, \mathrm{d}\omega_2, \mathrm{d}\omega_3) = \mathbb{P}'(\mathrm{d}\omega_1|\omega_3)\mathbb{P}'(\mathrm{d}\omega_2|\omega_3)\mathbb{P}'_{\bar{B}}(\mathrm{d}\omega_3),$$

such that, if we denote $(\omega_1, \omega_2, \omega_3) \in \Omega'$, we have that the process

$$\omega_1(\cdot) \sim X_\cdot, \quad \omega_2(\cdot) \sim X'_\cdot, \quad \omega_3(\cdot) \sim \bar{B} \sim \bar{B}'.$$

In other words, we want that $\mathbb{P}'(\omega_3|\mathrm{d}\omega_1)\mathbb{P}'_{\bar{B}}(\mathrm{d}\omega_3)$ is the law of the weak solution $(X, \bar{B}) \in C^0(\mathbb{R}_+, \mathbb{R}^m) \times C^0(\mathbb{R}_+, \mathbb{R}^n)$ and that $\mathbb{P}'(\omega_2|\mathrm{d}\omega_1)\mathbb{P}'_{\bar{B}}(\mathrm{d}\omega_3)$ is the law of the weak solution $(X', \bar{B}') \in C^0(\mathbb{R}_+, \mathbb{R}^m) \times C^0(\mathbb{R}_+, \mathbb{R}^n)$, i.e. we built a probability space where the weak solutions $(X, \bar{B})$ and $(X', \bar{B}')$ are driven by the same Brownian motion $\bar{B} = \bar{B}'$.

On this new probability, ω_3 is a Brownian motion and (ω_1, ω_3) and (ω_2, ω_3) are two weak solution to the SDE (μ, σ) with initial condition $\omega_1(0) = \omega_2(0) \sim Y$.

Since (μ, σ) has the pathwise uniqueness property then the processes ω_1 is indistinguishable from the process ω_2 (obviously with respect to the probability $\mathbb{P}'$), i.e.

$$\omega_1(\cdot) = \omega_2(\cdot),$$

almost surely. This means that the processes $\omega_1(\cdot) \sim \omega_2(\cdot)$ has the same law but since $\omega_1(\cdot) \sim X_\cdot$ and $\omega_2(\cdot) \sim X'_\cdot$ we conclude that $X_\cdot \sim X'_\cdot$. □

We return to the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}\bar{B}_t,$$

discussed in Section 6.3.2. In Theorem 6.22 we prove that (when b has at most linearly growth at infinity) then there exists a weak solution. We prove now a result on the uniqueness (in law) of the previous weak solution.

Theorem 6.24. *All the weak solutions $(X,\bar{B})$ of the SDE*

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}\bar{B}_t, \qquad t \geqslant 0, \tag{6.15}$$

satisfying for all $\tau \geqslant 0$

$$\int_0^\tau |b(X_s)|^2 \mathrm{d}s < \infty, \qquad a.s., \tag{6.16}$$

have the same law.

Remark 6.25. Under the hypotheses of Theorem 6.24, we do not require that b has at most linear growth at infinity but that $\int_0^\tau |b(X_s)|^2 \mathrm{d}s < \infty$ almost surely. If b has at most linearly growth at infinity then

$$\int_0^\tau |b(X_s)|^2 \mathrm{d}s \leqslant K \int_0^\tau (1+|X_s|)^2 \mathrm{d}s \leqslant 2K\tau \left(1 + \sup_{s\leqslant\tau} |X_s|^2\right),$$

which is always bounded since the process X_t is continuous.

Proof of Theorem 6.24. Let (X,B) any weak solution of (6.15) satisfying (6.16). Define the increasing sequence of stopping times $(T_n)_{n\geqslant 1}$ as

$$T_n = \inf\left\{t \geqslant 0 \colon \frac{1}{2}\int_0^t |b(X_s)|^2 \mathrm{d}s \geqslant n\right\},$$

and note that (6.16) implies $T_n \to \infty$ almost surely. Now consider $(Z_t)_{t\geqslant 0}$ as in eq. (6.12) above and observe that the process $(Q_t)_{t\geqslant 0}$ defined as $Q_t = Z_t^{-1}$ satisfies

$$\begin{aligned} Q_{t\wedge T_n} &:= Z_{t\wedge T_n}^{-1} = \exp\left(-\int_0^{t\wedge T_n} b(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^{t\wedge T_n} |b(X_s)|^2 \mathrm{d}s\right) \\ &= \exp\left(-\int_0^{t\wedge T_n} b(X_s)\mathrm{d}B_s - \frac{1}{2}\int_0^{t\wedge T_n} |b(X_s)|^2 \mathrm{d}s\right). \end{aligned}$$

Due to the presence of the stopping time, the Novikov's criterion is trivially satisfied, and we can define the measure $\mathbb{Q}^{(n)}$ such that $\mathrm{d}\mathbb{Q}^{(n)}|_{\mathcal{F}_t} = Q_{t\wedge T_n}\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$ for all $t \geqslant 0$ and under which

$$\tilde{B}_t^{(n)} = B_t + \int_0^{t\wedge T_n} b(X_s)\mathrm{d}s$$

is a Brownian motion. However by the SDE (6.15) we have $\tilde{B}_{t\wedge T_n}^{(n)} = X_{t\wedge T_n}$ so indeed $(X_t)_{t\geqslant 0}$ is a $\mathbb{Q}$-Brownian motion in the random interval $[0,T_n]$. As a consequence, for any $\tau \geqslant 0$ and any $\mathbb{1}_A(X,B) \in \mathcal{F}_\tau$ we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A(X,B)\mathbb{1}_{\tau\leqslant T_n}] \\ =\ & \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\tau\leqslant T_n}\mathbb{1}_A(X,B)Q_\tau^{-1}] \\ =\ & \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\tau\leqslant T_n}\mathbb{1}_A(X,B)\exp\left(\int_0^\tau b(X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^\tau |b(X_s)|^2\mathrm{d}s\right)\right] \\ =\ & \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\tau\leqslant T_n}\mathbb{1}_A\left(\tilde{B}_t^{(n)}, \tilde{B}_t^{(n)} - \int_0^t b(\tilde{B}_s^{(n)})\mathrm{d}s\right)\exp\left(\int_0^\tau b(\tilde{B}_s^{(n)})\mathrm{d}\tilde{B}_s^{(n)} - \frac{1}{2}\int_0^\tau |b(\tilde{B}_s^{(n)})|^2\mathrm{d}s\right)\right]. \end{aligned}$$

Since $\tilde{B}_t^{(n)}$ is a $\mathbb{Q}$-Brownian motion and B is a $\mathbb{P}$-Brownian motion we get that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\tau\leqslant T_n}\mathbb{1}_A\left(\tilde{B}_t^{(n)}, \tilde{B}_t^{(n)} - \int_0^t b(\tilde{B}_s^{(n)})\mathrm{d}s\right)\exp\left(\int_0^\tau b(\tilde{B}_s^{(n)})\mathrm{d}\tilde{B}_s^{(n)} - \frac{1}{2}\int_0^\tau |b(\tilde{B}_s^{(n)})|^2\mathrm{d}s\right)\right] \\ =\ & \mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{T\leqslant S_n(B)}\mathbb{1}_A\left(B, B - \int_0^\cdot b(B)\mathrm{d}s\right)\exp\left(\int_0^T b(B_s)\mathrm{d}B_s - \frac{1}{2}\int_0^T |b(B_s)|^2\mathrm{d}s\right)\right] = \mathbb{E}[h_A(B)], \end{aligned}$$

where $S_n(B) = \inf\left\{t \geqslant 0 \colon \frac{1}{2}\int_0^t |b(B_s)|^2 \mathrm{d}s \geqslant n\right\}$ and $h_A \colon C(\mathbb{R}_+; \mathbb{R}^n) \to \mathbb{R}_+$ is a suitable measurable function depending on A (and on τ). Since the previous proof holds for any weak solution (X, B), if we consider two weak solutions $(X^{(i)}, B^{(i)})$ $i = 1, 2$ we

$$\mathbb{E}\left[\mathbb{1}_A(X^{(1)}, B^{(1)})\mathbb{1}_{\tau \leqslant T_n^{(1)}}\right] = \mathbb{E}[h_A(B^{(1)})] = \mathbb{E}[h_A(B^{(2)})] = \mathbb{E}\left[\mathbb{1}_A(X^{(2)}, B^{(2)})\mathbb{1}_{\tau \leqslant T_n^{(2)}}\right],$$

where $T_n^{(i)} = S_n(X^{(i)})$ for $i = 1, 2$. Letting $n \to \infty$ and using (6.16) to prove that $T_n^{(i)} \to \infty$ a.s. for $i = 1, 2$, we deduce by dominated convergence that

$$\mathbb{E}[\mathbb{1}_A(X^{(1)}, B^{(1)})] = \mathbb{E}[\mathbb{1}_A(X^{(2)}, B^{(2)})].$$

Since $\tau \geqslant 0$ is arbitrary this equality holds for all $A \in \mathcal{B}(C(\mathbb{R}_+; \mathbb{R}^m))$ and therefore we conclude that $\mathrm{Law}(X^{(1)}, B^{(1)}) = \mathrm{Law}(X^{(2)}, B^{(2)})$ (as measures on $C(\mathbb{R}_+; \mathbb{R}^m)$). $\square$

Chapter 7

Local (in time) solutions of SDEs, Markov property, and relation with PDEs

7.1 Local (in time) solution to SDEs and explosion

7.1.1 Local existence and uniqueness

Definition 7.1. *Under the same hypotheses and notation of Section 6.1, we say that the continuous process X_t is a* ***local strong solution to the SDE (μ,σ) driven by the Brownian motion $\bar{B}$ with initial condition Y, till the stopping time T*** *if X_t is adapted with respect to the filtration $\sigma(Y,\mathcal{F}_t)$ and if, for any $t \geqslant 0$ and $k=1,\ldots,m$, we have*

$$X_t^{k,T} = X_{t\wedge T}^k = Y^k + \int_0^{T\wedge t} \mu^k(s,X_s)\mathrm{d}s + \sum_{j=1}^n \int_0^{T\wedge t} \sigma_j^k(s,X_s)\mathrm{d}B_s^j.$$

Definition 7.2. *Under the same hypotheses and notation of Section 6.3, we say that the pair of continuous process $(X,\bar{B})$ (adapted to the filtration $\{\mathcal{J}_t\}_{t\in\mathbb{R}_+}$) is a* ***local weak solution to the SDE (μ,σ) with initial condition Y, till the stopping time T*** *if $\bar{B}$ is adapted with respect to the filtration $\sigma(Y,\mathcal{F}_t)$ and if, for any $t\geqslant 0$ and $k=1,\ldots,m$, we have*

$$X_t^{k,T} = X_{t\wedge T}^k = Y^k + \int_0^{T\wedge t} \mu^k(s,X_s)\mathrm{d}s + \sum_{j=1}^n \int_0^{T\wedge t} \sigma_j^k(s,X_s)\mathrm{d}B_s^j.$$

Remark 7.3. We can extend the notion of pathwise uniqueness and uniqueness in law (till a stopping T) to the case of local strong and weak solutions.

Definition 7.4. *We say that the function (μ,σ) satisfies the local assumption A, if for any bounded closed set $U\subset\mathbb{R}^m$ there is a constant K_U such that, for any $t\geqslant 0$, $k=1,\ldots,m$, $j=1,\ldots,n$ and $x,y\in U$ we have*

$$|\mu^k(t,x)|\leqslant K_U(1+|x|),\quad |\sigma_j^k(t,x)|\leqslant K_U(1+|x|),$$

$$|\mu^k(t,x)-\mu^k(t,y)|\leqslant K_U|x-y|,\quad |\sigma_j^k(t,x)-\sigma_j^k(t,y)|\leqslant K_U|x-y|.$$

Theorem 7.5. *Suppose that the SDE (μ,σ) satisfies the local assumptions A, then, for any random variable $Y\in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$, the SDE (μ,σ) driven by the Brownian motion $\bar{B}$ with initial condition Y admits a local strong solution X_t till any stopping times T_U of the form, for any U open bounded set,*

$$T_U^X = \inf\{t\geqslant 0, X_t\in U^c\}. \tag{7.1}$$

Furthermore, under the previous hypotheses on (μ,σ), the solution is pathwise unique.

Remark 7.6. The strong solution X_t built in Theorem 7.5 does not depend on the stopping time T_U^X. The statement of the theorem says that there is a continuous stochastic process X_t taking values in $\mathbb{R}^m\cup\{\infty\}$ such that X_t is a strong local solution to the SDE (μ,σ) till any stopping time T_U^X.

Proof of Theorem 7.5. Fix U bounded open set, and define the map

$$S_U(X_{U,\cdot})^k(t) = Y^k + \int_0^{T_U^X \wedge t} \mu^k(s, X_{U,s}) \mathrm{d}s + \sum_{j=1}^n \int_0^{T_U^X \wedge t} \sigma_j^k(s, X_{U,s}) \mathrm{d}B_s^j.$$

If we define the set

$$\mathcal{X}_\lambda^Y := \{X \in \mathcal{X}_\lambda^Y, X_0 = Y\}.$$

Using the same methods of the proof of Theorem 6.8 we get that

$$\|S_U(X_{U,\cdot})\|_{\lambda'} \leqslant m\left(\|Y\|_{L^2(\Omega)} + \left[\sqrt{\frac{nK_U^2}{\lambda'}} + \sqrt{\frac{CK_U^2\pi}{\lambda'}}\right](1 + \|X_{U,\cdot}\|_{\lambda'})\right),$$

and also for any $X'_{U,\cdot}$

$$\|S_U(X_{U,\cdot}) - S_U(X'_{U,\cdot})\|_{\lambda'} \leqslant \left(\frac{m\sqrt{n\,K_U^2} + m\sqrt{\pi^2\,K_U^2 C}}{\sqrt{2\lambda'}}\right)\|X_{U,\cdot} - X'_{U,\cdot}\|_{\lambda'}.$$

We can then apply Banach fix point theorem and we obtain the existence and uniqueness of a process $X_{U,t}$ (which is such that $X_{U,T_U^{X_U} \wedge t} = X_{U,t}$) for which $S_U(X_U) = X_U$, which is equivalent to say that $X_{U,t}$ is a local strong solution to the SDE (μ, σ) till the stopping times T_U^X.

In order to prove that the process $X_{U,t}$ does not depend on U, or more precisely that there is a process X such that

$$X_t^{T_U^X} = X_{t \wedge T_U^X} = X_{U,t},$$

it is enough that if X_U is a solution till the stopping time $T_U^{X_U}$ and $X'_{U'}$ is a solution till the stopping time $T_{U'}^{X'_{U'}}$ we have

$$X_{U,T_{U'}^{X'_{U'}} \wedge t} = X_{U',T_U^{X_U} \wedge t},$$

almost surely. Indeed if X_U is a solution till the stopping time $T_U^{X_U}$ and $X'_{U'}$ is a solution till the stopping time $T_{U'}^{X'_{U'}}$ then both X_U and $X'_{U'}$ is solution to the stopping time

$$T_{U,U'} = T_U^{X_U} \wedge T_{U'}^{X_{U'}}.$$

Consider the process $Z_t^{U,U'} = X_{U,t}^{T_{U,U'}} - {X'_{U,t}}^{T_{U,U'}} = X_{U,t}^{T_{U'}^{X_{U'}}} - {X'_{U',t}}^{T_U^{X_U}}$, then using the same reasoning of the proof of Theorem 6.16

$$\mathbb{E}[|Z_t^{U,U'}|^2] \leqslant 2m(K + nK^2)\int_0^t \mathbb{E}[|Z_s^{U,U'}|^2] \mathrm{d}s,$$

and so, by Gronwall inequality, $Z_t^{U,U'}$ is indistinguishable from 0. This proves that

$$X_{U,t}^{T_{U'}^{X_{U'}}} = {X'_{U',t}}^{T_U^{X_U}},$$

almost surely. A consequence of the previous proof is that

$$T_{U,U'} = \inf\{t \geqslant 0, X_t \notin U \cap U'\},$$

and so

$$X_{U,t}^{T_{U'}^{X_{U'}}} = {X'_{U',t}}^{T_U^{X_U}} = X_{U \cap U',t}, \tag{7.2}$$

(i.e. any solution process $X_{U,t}^{T_{U'}^{X_{U'}}}$ restricted to any open subset $\tilde{U} \subset U$ of U is equal $X_{\tilde{U},t}$ which is the solution process of the equation restricted to the subset $\tilde{U}$).

In particular, let $B_N = \{x \in \mathbb{R}^m, |x| < N\}$ then we have

$$X_{B_N,t}^{T_{B_{N'}}} = X_{B_{N'},t},$$

if $N' \leqslant N$. Since the sequence T_{B_N} is increasing in N, the following process is well defined

$$X_t(\omega) := \begin{cases} \lim_{N\to+\infty} X^{T_{B_N}}_{B_N,t}(\omega), & \text{if } t \leqslant \lim_{N\to+\infty} T_{B_N}(\omega), \\ \infty, & \text{otherwise.} \end{cases}$$

The process X_t satisfies the properties of the thesis of Theorem 7.5. Indeed, if U is a bounded open set, there is $N \geqslant 0$ such that $U \subset B_N$. Let $(X_{U,t}, B)$ the solution to the equation (σ, μ) stopped in the set U, then by equality (7.2), we have

$$X_t^{T_U^X} = X_{U \cap B_N, t} = X_{U,t}. \qquad \square$$

Definition 7.7. *Under the hypotheses and the notation of Theorem 7.5, we call the explosion time of the SDE (μ, σ) with initial condition Y the stopping time*

$$E_Y := \lim_{N\to+\infty} T^{B_N}.$$

Remark 7.8. From the proof of Theorem 7.5, it is clear that the definition of T_Y^e does not depend on the increasing sequence $B_N = \{y \in \mathbb{R}^m, |y| > N\}$ of open sets converging to all $\mathbb{R}^m$. Indeed T_Y^e can also be defined as

$$E_Y = \inf\{t \geqslant 0, X_t = \infty\}.$$

Remark 7.9. From the proof of Theorem 7.5, it is clear that $\mathbb{P}(E_Y = 0) = 0$.

7.1.2 Explosion time and Lyapunov function

First of all we introduce the operator $\mathcal{L}_t: C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}) \to L^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})$ given by

$$\mathcal{L}_t(f)(t,x) = \sum_{k=1}^m \mu^k(t,x)\partial_{x^k} f(t,x) + \frac{1}{2}\sum_{k,k'=1}^m \sum_{j=1}^n \sigma_j^k(t,x)\sigma_j^{k'}(t,x)\partial^2_{x^k x^{k'}} f(t,x).$$

Lemma 7.10. *Let $(X, \bar{B})$ be a local weak solution to the SDE (μ, σ) till the stopping time T then, for any $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})$ we have*

$$\mathrm{d}f(t, X_t^T) = (\partial_t f(t, X_t^T) + \mathcal{L}_t f(t, X_t^T))\mathrm{d}t + \sum_{j=1}^n \left(\sum_{k=1}^m \sigma_j^k(t, X_t^T)\partial_{x^k} f(t, X_t^T) \right) \mathrm{d}B_t^{j,T}.$$

Proof. The proof is a simple application of Ito formula. $\square$

Definition 7.11. *Let $V: \mathbb{R}^m \to \mathbb{R}_+$ be a positive $C^2(\mathbb{R}^m, \mathbb{R})$ function, we say that V is a Lyapunov function for the SDE (μ, σ) if*

1. *$\lim_{x\to+\infty} V(x) = +\infty$;*
2. *there is $\lambda, A \in \mathbb{R}$ such that*

$$\mathcal{L}_t(V)(x) \leqslant \lambda V(x) + A.$$

Theorem 7.12. *Suppose the function V is a Lyapunov function for the SDE (μ, σ). Suppose that the SDE (μ, σ) satisfies local assumption A, it has a (local in time) weak solution $(X, \bar{B})$ with initial condition Y, such that $\mathbb{E}[V(Y)] < +\infty$, and explosion time E_Y. Then $E_Y < +\infty$ almost surely.*

Proof. In order to prove the theorem, it is enough to prove that for any $t > 0$ then $\mathbb{P}(E_Y \leqslant t) = 0$.

Consider the sequence of stopping times $T_N = \inf\{t \geqslant 0, |X_t| \geqslant N\}$, applying Ito formula to the process $e^{-\lambda t} V(X_t)$ we get

$$\begin{aligned} e^{-\lambda t} V(X_t^{T_N}) - V(Y) = & \int_0^{T_N \wedge t} (e^{-\lambda s}\mathcal{L}_s V(X_s) - \lambda e^{-\lambda s} V(X_s))\mathrm{d}s \\ & + \sum_{j=1}^n \int_0^{T_N \wedge t} \left(\sum_{k=1}^m e^{-\lambda s}\sigma_j^k(s, X_s)\partial_{x^k} V(X_s) \right) \mathrm{d}B_s^j. \end{aligned}$$

When $T_N>0$ we have that $|X_{t\wedge T_N}|\leqslant N$. This implies that (on the set $T_N>0$) $\sigma_j^k(s,X_s)\partial_{x^k}V(X_s)$ is a bounded process, and thus $\int_0^{T_N\wedge t}(\sum_{k=1}^m \sigma_j^k(s,X_s)\partial_{x^k}V(X_s))\mathrm{d}B_s^j$ is a $\mathcal{M}_c^2$ martingale. Furthermore the process $(e^{-\lambda s}\mathcal{L}_sV(X_s)-\lambda e^{-\lambda s}V(X_s))$ is bounded and we have also

$$V(X_t^{T_N})\leqslant\left(\left(\max_{|x|\leqslant N}V(x)\right)+V(Y)\right)\in L^1(\Omega).$$

We can take the expectation at both side of the integral obtaining

$$\mathbb{E}[e^{-\lambda(T_N\wedge t)}V(X_t^{T_N})]-\mathbb{E}[V(Y)]=\mathbb{E}\left[\int_0^{T_N\wedge t}(e^{-\lambda s}\mathcal{L}_sV(X_s)-\lambda e^{-\lambda s}V(X_s))\mathrm{d}s\right]\leqslant A\left(\frac{1-e^{-\lambda t}}{\lambda}\right)\leqslant C$$

since $(e^{-\lambda s}\mathcal{L}_sV(X_s)-\lambda e^{-\lambda s}V(X_s))\leqslant e^{-\lambda s}A$ being V a Lyapunov function, and $C\in\mathbb{R}_+$. Thus we get

$$\begin{aligned}(\mathbb{E}[V(Y)]+C) &\geqslant \mathbb{E}[e^{-\lambda(T_N\wedge t)}V(X_t^{T_N})]\geqslant\mathbb{E}[e^{-\lambda(T_N\wedge t)}V(X_{T_N})\mathbb{I}_{T_N<t}] &(7.3)\\ &\geqslant e^{-|\lambda|t}\left(\inf_{|x|=N}V(x)\right)\mathbb{P}(0<T_N<t)-e^{-|\lambda|t}\mathbb{E}[V(Y)]\mathbb{P}(T_N=0) &(7.4)\end{aligned}$$

On the other $T_N\to E_Y$ almost surely and so, by Remark 7.9,

$$\mathbb{P}(T_N=0)\to\mathbb{P}(E_Y=0)=0.$$

We have also

$$\mathbb{P}(0<T_N<t)\to\mathbb{P}(0\leqslant E_Y\leqslant t)=\mathbb{P}(E_Y\leqslant t).$$

On the other hand since $\lim_{x\to\infty}V(x)=+\infty$, we have $(\inf_{|x|=N}V(x))\to+\infty$ as $N\to+\infty$. Finally we get

$$e^{|\lambda|t}(\mathbb{E}[V(Y)]+C)\geqslant\limsup_{N\to+\infty}\left(\left(\inf_{|x|=N}V(x)\right)\mathbb{P}(0<T_N<t)\right)$$

we need to have $\mathbb{P}(0<T_N<t)\to0$, and thus $\mathbb{P}(E_Y\leqslant t)=0$. □

Corollary 7.13. *Under the hypotheses of Theorem 7.12, we have that, for any $t\geqslant0$,*

$$\mathbb{E}[V(X_t)]\leqslant e^{\lambda t}\left(\mathbb{E}[V(Y)]+\frac{A(1-e^{-\lambda t})}{\lambda}\right). \tag{7.5}$$

Proof. Under the previous hypotheses, by the proof of Theorem 7.12, more precisely inequality (7.3), we get

$$\mathbb{E}[e^{-\lambda(T_N\wedge t)}V(X_t^{T_N})]\leqslant\left(\mathbb{E}[V(Y)]+\frac{A(1-e^{-\lambda t})}{\lambda}\right). \tag{7.6}$$

Since, by Theorem 7.12, $T_N:=\inf\{t\geqslant0,|X_t|\geqslant N\}\to+\infty$ almost surely (and thus $T_N\wedge t\to t$ almost surely), we get the thesis of corollary, by taking the limit of inequality (7.6) as $N\to+\infty$. □

An important consequence of the previous corollary is the following one.

Theorem 7.14. *Suppose that the SDE (μ,σ) satisfies local assumption A and that there is $K>0$ for which*

$$|\mu^k(x)|\leqslant K(1+|x|),\quad |\sigma_j^k|\leqslant K(1+|x|). \tag{7.7}$$

Suppose that there is $p>0$ such that $\mathbb{E}[|Y|^p]$, then, for any $t\geqslant0$, we have $\mathbb{E}[|X_t|^p]<+\infty$.

Proof. By Corollary 7.13 it is enough to prove that

$$V(x)=(1+|x|^2)^{p/2}=\left(1+\sum_{k=1}^m(x^k)^2\right)^{\frac{p}{2}}$$

is a Lyapunov function for the SDE (μ,σ) when the linear growth at infinity of the coefficients holds. We have

$$\begin{aligned}\mathcal{L}_t(V(x)) &= p\left(1+\sum_{k=1}^m (x^k)^2\right)^{\frac{p}{2}-1}\left(\sum_{k=1}^m \mu^k(t,x)x^k+\frac{1}{2}\sum_{k=1}^m\left(\sum_{j=1}^n \sigma_j^k(t,x)\sigma_j^k(t,x)\right)\right)\\ &\quad +p(p-2)\left(1+\sum_{k=1}^m (x^k)^2\right)^{\frac{p}{2}-2}\left(\frac{1}{2}\sum_{k,h=1}^m\left(\sum_{j=1}^n \sigma_j^k(t,x)\sigma_j^k(t,x)x^kx^h\right)\right).\end{aligned}$$

Thus, using inequality (7.7), we get

$$\begin{aligned}\mathcal{L}_t(V(x)) &\leqslant p(1+|x|^2)^{p/2-1}(Km(1+|x|)|x|+K^2mn|x|^2)\\ &\quad +|p(p-2)|K^2(1+|x|^2)^{p/2-2}(1+|x|^2)|x|^2\\ &\leqslant C[(1+|x|^2)^{p/2-1}(1+|x|^2)+(1+|x|^2)^{p/2-2}(1+|x|^2)]\\ &\leqslant CV(x),\end{aligned}\tag{7.8}$$

where $C>0$ is a suitable constant dependent on p, K, m, and n. □

Corollary 7.15. *Under the hypotheses of Theorem 7.14, for any $p>0$ there is a constant $C_p>0$ such that*

$$\mathbb{E}[|X_t|^p]\leqslant C_pe^{C_pt}(1+\mathbb{E}[|Y|^p]).$$

Proof. The Corollary is a consequence of Corollary 7.13, Theorem 7.14 and inequality (7.8). □

The presence of Lyapunov functions not only permits to obtain some better linear bounds on the moments of the process X_t when the growth of the coefficient is linear, but it also allows us to study some SDE with coefficient with superlinear growth of the coefficients.

Consider $m=n=1$ and let $\mu(t,x)=\mu(x)=-x^{2k-1}$, for some $k\in\mathbb{N}$, $k\geqslant 1$, and $\sigma(t,x)=\sigma(x)=1$, i.e. the SDE

$$\mathrm{d}X_t=-X_t^{2k-1}\mathrm{d}t+\mathrm{d}B_t.\tag{7.9}$$

Since $\mu,\sigma\in C^\infty(\mathbb{R},\mathbb{R})$, by Lagrange theorem, the SDE (μ,σ) satisfies local assumption A. In this case the operator $\mathcal{L}_t=\mathcal{L}$ is

$$\mathcal{L}(f)(t,x)=-x^{2k-1}\partial_xf(t,x)+\frac{1}{2}\partial_x^2f(t,x).$$

Consider

$$V(x)=|x|^{2p}$$

for $p\geqslant 1$, then we have

$$\begin{aligned}\mathcal{L}(V)(x) &= -2px^{2k-1}\mathrm{sign}(x)|x|^{2p-1}+p(2p-2)|x|^{2p-2}=-2p\,|x|^{2k+2p-1}+p(2p-2)|x|^{2p-2}\\ &\leqslant p(2p-2)|x|^{2p-2}\leqslant p(2p-2)V(x)+p(2p-2).\end{aligned}$$

So $|x|^{2p}$ are Lyapunov function also in the case where $\mu(x)=-x^{2k-1}$ and $\sigma(x)=1$.

We can get better estimate for the expected value of X_t if we consider

$$V(x)=e^{(1+x^2)^{k-\varepsilon}}.$$

for some $0<\varepsilon<k$. In this case, we have

$$\begin{aligned}\mathcal{L}(V)(x) &= -(k-\varepsilon)x^{2k}(1+x^2)^{k-1-\varepsilon}e^{(1+x^2)^{2k-1-\varepsilon}}+\frac{1}{2}\left(\frac{k}{2}-\varepsilon\right)^2x^2(1+x^2)^{2k-2-2\varepsilon}e^{(1+x^2)^{2k-1-\varepsilon}}\\ &\quad +\frac{1}{2}\left(\frac{k}{2}-\varepsilon\right)\left(\frac{k}{2}-1-\varepsilon\right)x^2(1+x^2)^{k/2-2-\varepsilon}e^{(1+x^2)^{2k-1-\varepsilon}}.\end{aligned}$$

So, for any $\lambda\in\mathbb{R}$, we get that

$$\mathcal{L}(V)(x)-\lambda V(x)\leqslant\{-(k-\varepsilon)x^{4k-2-2\varepsilon}+C_{k,\varepsilon}(1+x^{4k-2-4\varepsilon})\}e^{(1+x^2)^{2k-1-\varepsilon}}.$$

Since $\lim_{|x|\to+\infty}\{-(k-\varepsilon)x^{4k-2-2\varepsilon}+C_{k,\varepsilon}(1+x^{4k-2-4\varepsilon})\}e^{(1+x^2)^{2k-1-\varepsilon}}=-\infty$, there is a constant $A_\lambda\in\mathbb{R}_+$ such that

$$\{-(k-\varepsilon)x^{4k-2-2\varepsilon}+C_{k,\varepsilon,\lambda}(1+x^{4k-2-4\varepsilon})\}e^{(1+x^2)^{2k-1-\varepsilon}}\leqslant A_{k,\varepsilon,\lambda}.$$

This means that for any $\lambda\in\mathbb{R}_+$ there is

$$\mathcal{L}(V)(x)\leqslant\lambda V(x)+A_{k,\varepsilon,\lambda}.$$

For example if the initial condition $Y=y\in\mathbb{R}$ is deterministic Corollary 7.13 and the previous computation implies that

$$\mathbb{E}[\exp((1+X_t^{y,2})^{k-\varepsilon})]\leqslant\left(\exp((1+y^2)^{k-\varepsilon})+\frac{A_{k,\varepsilon,\lambda}(1-e^{-\lambda t})}{\lambda}\right),$$

where X_t^y is the solution to the equation (7.9) with initial condition $X_0^y=y\in\mathbb{R}$.

Definition 7.16. *Let $V:\mathbb{R}^m\to\mathbb{R}_+$ be a positive $C^2(\mathbb{R}^m,\mathbb{R})$ function, we say that V is a anti-Lyapunov function if*

1. $\sup_{x\in\mathbb{R}^m}V(x)<+\infty$;
2. *There is $\lambda>0$ such that*

$$\mathcal{L}_t(V)(x)\geqslant\lambda V(x).$$

Theorem 7.17. *Suppose that the SDE (μ,σ) (satisfying local assumption A) admits an anti-Lyapunov function V then for any $Y=y\in\mathbb{R}^m$ (deterministic) such that $V(y)>0$ we have*

$$\mathbb{P}(E_y<+\infty)>0.$$

Proof. Since $V(y)$ is strictly positive, there is $\tau>0$ such that

$$V(y)>e^{-\lambda\tau}\left(\sup_{x\in\mathbb{R}^m}V(x)\right). \tag{7.10}$$

Define the stopping times $T_N=\inf\{t\geqslant0,|X_t|\geqslant N\}$ as usual. Since there is $N_0>0$ such that $B_{N_0}=\{|y|<N_0\}$ for which $y\in B_{N_0}$ then $T_N>0$ almost surely for $N\geqslant N_0$. Following similar computations to the ones done in the proof of Theorem 7.12, we get that

$$V(y)\leqslant e^{-\lambda\tau}\mathbb{E}[V(X_\tau^{T_N})]\leqslant e^{-\lambda\tau}\left(\sup_{x\in\mathbb{R}^m}V(x)\right)\mathbb{P}(T_N>\tau)+\left(\sup_{x\in\mathbb{R}^m}V(x)\right)\mathbb{P}(T_N\leqslant\tau).$$

If $\mathbb{P}(T_N\leqslant\tau)\to0$ as $N\to+\infty$, we get that, for any $\varepsilon>0$, for any $N\geqslant N_\varepsilon$ big enough $\mathbb{P}(T_N>\tau)>1-\varepsilon$ and thus

$$V(y)\leqslant e^{-\lambda\tau}\left(\sup_{x\in\mathbb{R}^m}V(x)\right)(1-\varepsilon)+\left(\sup_{x\in\mathbb{R}^m}V(x)\right)\mathbb{P}(T_N\leqslant\tau)\to e^{-\lambda\tau}\left(\sup_{x\in\mathbb{R}^m}V(x)\right)(1-\varepsilon).$$

For ε small enough, the previous inequality contradicts inequality (7.10). This means that $\mathbb{P}(T_N\leqslant\tau)\to C>0$. On the other hand, for what we said in the proof of Theorem 7.12, $\mathbb{P}(T_N\leqslant\tau)\to\mathbb{P}(E_y\leqslant\tau)$ and the proof of the theorem is concluded. □

We now propose an example of SDE with explosion time $\mathbb{P}(E_y<+\infty)>0$ for some (deterministic) initial condition $y\in\mathbb{R}^m$. Let $m=n=1$ and consider the (additive noise) SDE

$$\mathrm{d}X_t=+X_t^3\mathrm{d}t+\mathrm{d}B_t,\qquad X_0=y\in\mathbb{R},$$

namely $\mu(x)=x^3$ and $\sigma(x)=1$. Consider the function

$$V(x)=\frac{x^2}{x^2+1},\qquad x\in\mathbb{R},$$

which is strctly positive for $x \neq 0$. We now prove that $V(x)$ for the SDE $(\mu, \sigma) = (x^3, 1)$ and any initial condition $y \in \mathbb{R}$. We have that

$$\mathcal{L}_t(V) = \left(-\frac{2x^4}{(x^2+1)^2} + \frac{2x^4}{x^2+1}\right) + \left(\frac{1}{x^2+1} - \frac{5}{(x^2+1)^2} + \frac{8x^4}{(x^2+1)^3}\right) =$$

$$= \frac{2x^6 + 2x^4 - 3x^2 + 1}{(x^2+1)^3}.$$

Consider the polynomial $P(z) = 2z^3 + 2z^2 - 3z + 1$ for $z \geqslant 0$ (here $z = x^2$). We have

$$P'(z) = 6z^2 + 4z - 3$$

which has zeros in

$$z_{1,2} = \frac{-2 \pm \sqrt{4+18}}{6},$$

which implies that $P(z)$ has a minimum in $z = x^2 = \frac{-2+\sqrt{22}}{6}$. This implies that, for any $x \in \mathbb{R}$,

$$\mathcal{L}_t(V)(x) \geqslant \frac{P\left(\frac{-2+\sqrt{22}}{6}\right)}{(x^2+1)^3} = \frac{58 - 11\sqrt{22}}{27(x^2+1)} \geqslant \frac{6}{27(x^2+1)} > 0.$$

Since

$$\lim_{|x| \to +\infty} \mathcal{L}_t(V)(x) = 2$$

this implies that there is $C > 0$ such that

$$\mathcal{L}_t(V)(x) > C, \quad x \in \mathbb{R}.$$

Finally, since $V(x) \leqslant 1$, we get that there is $\lambda > 0$ for which

$$\mathcal{L}_t(V)(x) \geqslant \lambda V(x).$$

7.2 Markov property of the solutions to autonomous SDEs

7.2.1 Continuous dependence of solutions on (deterministic) initial condition

If the SDE (μ, σ) satisfies assumption A, we proved in Section 6.2, that there is a (unique) strong solution for any $Y \in L^2(\mathcal{F}_0)$. In other words, there is a map

$$F: \mathbb{R}_+ \times L^2(\mathcal{F}_0) \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R},$$

such that for any $\tau > 0$ the restriction of $F|_{[0,\tau]}: [0, \tau] \times L^0(\mathcal{F}_0) \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}$ is $\mathcal{B}([0, \tau]) \otimes \mathcal{B}(L^2(\mathcal{F}_0)) \otimes \mathcal{B}(C^0([0, t], \mathbb{R}^n))$ measurable (i.e. the map F is progressive), and such that the process

$$X_t = F(t, Y, \bar{B}_{[0,t]})$$

is the strong solution to the SDE (μ, σ) driven by the Brownian motion $\bar{B}$ and with initial condition $Y \in L^2(\mathcal{F}_0)$.

We restrict now to the case where the initial condition $Y = y \in \mathbb{R}^m$ is deterministic.

Theorem 7.18. *Let (μ, σ) be a SDE satisfying assumption A, then there is a function*

$$\bar{F}: \mathbb{R}_+ \times \mathbb{R}^m \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^m,$$

such that for any $\tau > 0$ the restriction of $F|_{[0,\tau]}: [0, \tau] \times L^0(\mathcal{F}_0) \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}$ is $\mathcal{B}([0, \tau]) \otimes \mathcal{B}(L^2(\mathcal{F}_0)) \otimes \mathcal{B}(C^0([0, t], \mathbb{R}^n))$ measurable (i.e. the map F is progressive), for it is continuous in the second component $\mathbb{R}^m$, and such that for any n-dimensional Brownian motion $\bar{B}$ and for any $y \in \mathbb{R}^m$ the process

$$X_t^y = \bar{F}(t, y, \bar{B}_{[0,t]})$$

is the unique strong solution to the SDE (μ, σ) driven by $\bar{B}$ and with initial condition $y \in \mathbb{R}^m$.

Remark 7.19. Consider two functions $\bar{F}: \mathbb{R}_+ \times \mathbb{R}^m \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^m$ and $\tilde{F}: \mathbb{R}_+ \times \mathbb{R}^m \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^m$ which are measurable with respect all variables and continuous with respect the first two variable. Let $\mathbb{P}_B(\mathrm{d}\gamma)$ the probability measure on the space $C^0(\mathbb{R}_+, \mathbb{R}^n)$ such that the paths $\gamma \in C^0(\mathbb{R}_+, \mathbb{R}^n)$ are Brownian motion (with respect to the probability $\mathbb{P}_B$), i.e. for each $t_1 < \cdots < t_n$, the random variables $\gamma(t_2) - \gamma(t_1), \ldots, \gamma(t_n) - \gamma(t_{n-1})$ are independent normal random variables with mean 0 and variance $t_2 - t_1, \ldots, t_n - t_{n-1}$. Suppose that there is a (measurable) set $\Gamma \subset C^0(\mathbb{R}_+, \mathbb{R}^n)$ such that $\mathbb{P}_B(\Gamma) = 1$ and

$$\bar{F}(t, y, \gamma) = \tilde{F}(t, y, \gamma),$$

for any $t \in \mathbb{R}_+$, $y \in \mathbb{R}^m$ and $\gamma \in \Gamma$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space and $B: \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then we have

$$\bar{F}(t, y, B_{[0,t]}(\omega)) = \tilde{F}(t, y, B_{[0,t]}(\omega))$$

for every $t, y \in \mathbb{R}_+ \times \mathbb{R}^m$ and almost surely with respect to $\omega \in \Omega$ (with respect to the probability $\mathbb{P}$).

In order to prove the previous theorem, we need the following result.

Theorem 7.20. (Kolmogorov continuity criterion) *Let*

$$X_t^y: \mathbb{R}_+ \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$$

be a continuous adapted stochastic process (with respect to $t \in \mathbb{R}_+$) taking values in $\mathbb{R}^m$ and depending on the parameter $y \in \mathbb{R}^m$, such that there is a $p > 1$ and a $\gamma > m$ for which for any $\tau > 0$ there is a constant $C_\tau > 0$ such that

$$\mathbb{E}\left[\sup_{t \leqslant \tau} |X_t^y - X_t^{y'}|^p\right] < C_\tau |y - y'|^\gamma.$$

*Then, there is a process $\tilde{X}_t^y: \mathbb{R}_+ \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ which is continuous adapted stochastic process (with respect to $t \in \mathbb{R}_+$) taking values in $\mathbb{R}^m$ and depending **continuously** on the parameter $y \in \mathbb{R}^m$, such that for any $y \in \mathbb{R}^m$ and $t \in \mathbb{R}_+$ we have*

$$X_t^y = \tilde{X}_t^y$$

almost surely.

Proof. The proof of this theorem can be found in Chapter 2 Theorem 2.9 of [4] or Chapter 1 Theorem 1.8.1 of [3]. □

Proof of Theorem 7.18. For simplicity of notation, we consider the case $m = n = 1$ (i.e. we have $\bar{B}_t = B_t^1 = B_t$ a one-dimensional Brownian motion and $X_t = X_t^1$ is a one-dimensional process). For this reason, we write $\mu(t, x) := \mu^1(t, x)$ and $\sigma(t, x) := \sigma_1^1(t, x)$. The general case is a simple generalization.

Fix the probability space

$$\Omega' = C^0(\mathbb{R}_+, \mathbb{R})$$

with the Borel σ-algebra. Let $\mathbb{P}$ the probability measure on Ω' such that the process

$$B_t(\omega) := \omega(t)$$

is a one-dimensional Brownian motion. Let $\mathcal{F}_t$ be the natural filtration of B_t. Let X_t^y be the (unique up to null sets) strong solution of the SDE (μ, σ) driven by the Brownian motion B_t and with initial condition $y \in \mathbb{R}^m = \mathbb{R}$. If $y, y' \in \mathbb{R}$ we write $Z_t^{y,y'} = X_t^y - X_t^{y'}$. With this notation we have, for any $p \geqslant 4$,

$$\begin{aligned}
\mathrm{d}|Z_t^{y,y'}|^{p/2} &= \frac{1}{2}(p-2)|Z_t^{y,y'}|^{p/2-1}\mathrm{d}(Z_t^{y,y'}) + \frac{(p-2)(p-4)}{4}|Z_t^{y,y'}|^{p/2-2}(\sigma(t, X_t^y) - \sigma(t, X_t^{y'}))^2\mathrm{d}t \\
&= \frac{(p-2)}{2}|Z_t^{y,y'}|^{p/2-1}(\mu(t, X_t^y) - \mu(t, X_t^{y'}))\,\mathrm{d}t \\
&\quad + \frac{(p-2)(p-4)}{4}|Z_t^{y,y'}|^{p/2-2}(\sigma(t, X_t^y) - \sigma(t, X_t^{y'}))\mathrm{d}t \\
&\quad + \frac{(p-2)}{2}|Z_t^{y,y'}|^{p/2-1}(\sigma(t, X_t^y) - \sigma(t, X_t^{y'}))\mathrm{d}B_t.
\end{aligned}$$

We recall that, since $|y|,|y'|\in L^\infty(\Omega)\subset L^p(\Omega)$ for every $p\geqslant 1$, by Theorem 7.14 we have that, for any $p\geqslant 4$ and any $y,y'\in\mathbb{R}$, $X_t^y, X_t^{y'}\in L^p(\Omega)$ and so $Z^{y,y'}$ is a L^p semimartingale. Thus, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leqslant\ell}|Z_t^{y,y'}|^p\right]\\
\leqslant\;& 4|y-y'|^p+4\mathbb{E}\left[\left|\sup_{t\leqslant\ell}\int_0^t\frac{(p-2)}{2}|Z_s^{y,y'}|^{p/2-1}(\mu(s,X_t^y)-\mu(s,X_t^{y'}))\mathrm{d}s\right|^2\right]\\
&+4\mathbb{E}\left[\left|\sup_{t\leqslant\ell}\int_0^t\frac{(p-2)(p-4)}{4}|Z_t^{y,y'}|^{p/2-2}(\sigma(t,X_t^y)-\sigma(t,X_t^{y'}))^2\right|^2\right]\\
&+4\mathbb{E}\left[\sup_{t\leqslant\ell}\left|\int_0^t\frac{(p-2)}{2}|Z_s^{y,y'}|^{p/2-1}(\sigma(t,X_s^y)-\sigma(t,X_s^{y'}))\mathrm{d}B_s\right|^2\right]\\
\leqslant\;& 4|y-y'|^p+4\frac{(p-2)}{2}K\tau\int_0^\ell\mathbb{E}[|Z_s^{y,y'}|^p]\mathrm{d}s+(p-2)(p-4)K^2\int_0^\ell\mathbb{E}[|Z_s^{y,y'}|^p]\mathrm{d}s\\
&+4\mathbb{E}\left[\int_0^\ell\left(\frac{(p-2)}{2}\right)^2|Z_s^{y,y'}|^{p-2}(\sigma(t,X_s^y)-\sigma(t,X_s^{y'}))^2\mathrm{d}s\right]\\
\leqslant\;& 4|y-y'|^p+C_{K,\tau,p}\int_0^\ell\mathbb{E}[|Z_s^{y,y'}|^p]\mathrm{d}s\leqslant 4|y-y'|^p+C_{K,\tau,p}\int_0^\ell\mathbb{E}\left[\sup_{s\leqslant t}|Z_s^{y,y'}|^p\right]\mathrm{d}t,
\end{aligned}$$

for a suitable constant $C_{K,\tau,p}>0$ depending on K, τ and p. If we denote by $f^{y,y'}(\ell):=\mathbb{E}[\sup_{t\leqslant\ell}|Z_t^{y,y'}|^p]$, then we get the integral inequality

$$f^{y,y'}(\ell)\leqslant 4|y-y'|^p+C_{K,\tau,p}\int_0^\ell f^{y,y'}(t)\mathrm{d}t,$$

and thus, by Grownall lemma, we obtain that, for any $\ell\leqslant\tau$,

$$f^{y,y'}(\ell)\leqslant 4|y-y'|^p\,e^{C_{K,\tau,p}t}.$$

This means that there is a constant $\tilde{C}_{K,\tau,p}>0$ such that

$$\mathbb{E}\left[\sup_{t\leqslant\tau}|X_t^y-X_t^{y'}|^p\right]=\mathbb{E}\left[\sup_{t\leqslant\tau}|Z_t^{y,y'}|^p\right]\leqslant\tilde{C}_{K,\tau,p}|y-y'|^p.$$

If we choose $p>m=1$, we can apply Theorem 7.20, i.e. there is a measurable map

$$\tilde{X}_t^y\colon\mathbb{R}_+\times\mathbb{R}^m\times\Omega'\to\mathbb{R}^m$$

which is predictable with respect to t, and continuous with respect to both $t\in\mathbb{R}_+$ and $y\in\mathbb{R}$ such that

$$X_t^y=\tilde{X}_t^y,$$

almost surely. Since $\tilde{X}_t^y$ is predictable with respect to the σ-algebra generated by the Brownian motion B_t and the (deterministic) initial condition $y\in\mathbb{R}$, and $\tilde{X}_t^y$ is almost surely equal to the strong solution X_t^y, the process $\tilde{X}_t^y$ is also a strong solution to the SDE (μ,σ) driven by Brownian motion B_t and initial condition $y\in\mathbb{R}$. Now we define

$$\bar{F}(t,y,\omega):=\tilde{X}_t^y(\omega).$$

The theorem is proved. □

Theorem 7.21. *Suppose that (μ,σ) satisfies the assumption A and let X_t be the strong solution to the SDE (μ,σ) driven by the Brownian motion $\bar{B}$ and with initial condition $Y\in L^2(\mathcal{F}_0)$. Then there is a set $\Omega_1\subset\Omega$ of full measure such that, for any $t\geqslant 0$, we have*

$$X_t(\omega)=\bar{F}(t,Y(\omega),B_{[0,t]}(\omega)),$$

almost surely, where $\bar{F}\colon\mathbb{R}_+\times\mathbb{R}^m\times C^0(\mathbb{R}_+,\mathbb{R}^n)\to\mathbb{R}^m$ is the map in the thesis of Theorem 7.18.

Proof. Since $\bar{F}(\cdot,\cdot,\bar{B}_{[0,t]})$ is predictable and continuous with respect the first two variables, the process $(t,\omega)\mapsto \bar{F}(t,Y(\omega),\bar{B}_{[0,t]}(\omega))$ is predictable and continuous with respect to the time. Furthermore, $\bar{F}(t,Y(\omega),\bar{B}_{[0,t]}(\omega))$ is measurable with respect to the σ-algebra $\sigma(\sigma(Y),\mathcal{F}^B_t)$ (i.e. the σ-algebra generated by Y and $\bar{B}_{[0,t]}$). If we are able to prove that $\bar{F}(t,Y(\omega),\bar{B}_{[0,t]}(\omega))$ satisfies the SDE (μ,σ) by the uniqueness of strong solution the theorem is proved.

By definition of $\bar{F}$ we have that

$$(\bar{F}(t,x,B_{[0,t]}))^k = x^k + \int_0^t \mu^k(s,\bar{F}(s,x,B_{[0,s]}))\mathrm{d}s + \sum_{j=1}^n \int_0^t \sigma^k_j(s,\bar{F}(s,x,B_{[0,s]}))\mathrm{d}B^j_s. \tag{7.11}$$

Consider the (measurable and adapted) maps

$$M^k\colon \mathbb{R}^m\times\mathbb{R}_+\times C^0(\mathbb{R}_+,\mathbb{R}^n)\to\mathbb{R},$$

$$S^k_j\colon \mathbb{R}^m\times\mathbb{R}_+\times C^0(\mathbb{R}_+,\mathbb{R}^n)\to\mathbb{R},$$

defined as

$$M^k(x,t,B_{[0,t]}) := \lim_{|\pi|\to 0}\sum_{t_\ell\in\pi^t} \mu^k(t_{\ell-1},\bar{F}(t_{\ell-1},x,B_{[0,t_{\ell-1}]}))(t_\ell - t_{\ell-1}),$$

and

$$S^k_j(x,t,B_{[0,t]}) := \lim_{|\pi|\to 0}\sum_{t_\ell\in\pi^t} \sigma^k_j(t_{\ell-1},\bar{F}(t_{\ell-1},x,B_{[0,t_{\ell-1}]}))(B_{t_\ell} - B_{t_{\ell-1}}).$$

By the definition of Riemann-Stieltjes and Ito integral we have that equation (7.11) implies

$$(\bar{F}(t,x,B_{[0,t]}))^k = x^k + M^k(x,t,B_{[0,t]}) + \sum_{j=1}^n S^k_j(x,t,B_{[0,t]}). \tag{7.12}$$

On the other hand, we have

$$\begin{aligned}\left(\int_0^t \mu^k(\bar{F}(s,Y,B_{[0,s]}))\mathrm{d}s\right)(\omega) &= \lim_{|\pi|\to 0}\sum_{t_\ell\in\pi^t}\mu^k(t_{\ell-1},\bar{F}(t_{\ell-1},Y(\omega),B_{[0,t_{\ell-1}]}(\omega)))(t_\ell - t_{\ell-1})\\ &= M^k(Y(\omega),t,B_{[0,t]}(\omega)),\end{aligned}$$

and similarly

$$\begin{aligned}\left(\int_0^t \sigma^k_j(\bar{F}(s,Y,B_{[0,s]}))\mathrm{d}B^j_s\right)(\omega) &= \lim_{|\pi|\to 0}\sum_{t_\ell\in\pi^t}\sigma^k_j(t_{\ell-1},\bar{F}(t_{\ell-1},Y(\omega),B_{[0,t_{\ell-1}]}(\omega)))(t_\ell - t_{\ell-1})\\ &= S^k_j(Y(\omega),t,B_{[0,t]}).\end{aligned}$$

Replacing x by $Y(\omega)$ in equation (7.11) and the previous expression for $M^k(Y(\omega),t,B_{[0,t]})$ and $S^k_j(Y(\omega),t,B_{[0,t]})$, we get

$$\begin{aligned}&(\bar{F}(t,Y(\omega),B_{[0,t]}(\omega)))^k\\ =\ & Y^k(\omega) + M^k(Y(\omega),t,B_{[0,t]}(\omega)) + \sum_{j=1}^n S^k_j(Y(\omega),t,B_{[0,t]}(\omega))\\ =\ & Y^k(\omega) + \left(\int_0^t \mu^k(\bar{F}(s,Y,B_{[0,s]}))\mathrm{d}s\right)(\omega) + \sum_{j=1}^n\left(\int_0^t \sigma^k_j(\bar{F}(s,Y,B_{[0,s]}))\mathrm{d}B^j_s\right)(\omega).\end{aligned}$$

Thus $(\bar{F}(t,Y(\omega),B_{[0,t]}(\omega)))^k$ is a strong solution to the SDE (μ,σ) driven by $\bar{B}$ and with initial condition Y, and by uniqueness of strong solution we have $\bar{F}(t,Y(\omega),B_{[0,t]}(\omega)) = X_t(\omega)$ for almost every $\omega\in\Omega$. □

7.2.2 Markov property of strong solutions

Definition 7.22. *An SDE (μ,σ) is called autonomous if*

$$\mu^k(t,x) = \mu^k(x),\quad \sigma^k_j(t,x) = \sigma^k_j(x),$$

i.e. the coefficients do not depend explicitly on the time $t \geqslant 0$.

Definition 7.23. *Let* X_t *be a* ($\mathbb{R}^m$)*-stochastic process and let* $\mathcal{F}_t^X = \sigma(X_s, s \leqslant t)$ *its natural filtration. We say that the process* X *is a Markov process if, for any (bounded) continuous function* G: $\mathbb{R}^{mk} \to \mathbb{R}$ *and any* $t \leqslant t_1 \leqslant \cdots \leqslant t_k \in \mathbb{R}_+$, *we have*

$$\mathbb{E}[G(X_{t_1}, \ldots, X_{t_k}) | \mathcal{F}_t] = \mathbb{E}[G(X_{t_1}, \ldots, X_{t_k}) | X_t].$$

For any $t \geqslant 0$, we consider the map Q_t: $C_b^0(\mathbb{R}^m, \mathbb{R}) \to C_b^0(\mathbb{R}^m, \mathbb{R})$ (where $C_b^0(\mathbb{R}^m, \mathbb{R})$ is the set of bounded continuous functions on $\mathbb{R}^m$) given by

$$Q_t(f)(x) := \mathbb{E}[f(\bar{F}(t, x, \bar{B}_{[0,t]}))], \qquad f \in C_b^0(\mathbb{R}^m, \mathbb{R}), x \in \mathbb{R}^m,$$

where $\bar{F}$: $\mathbb{R}_+ \times \mathbb{R}^m \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}$ is the function introduced in Theorem 7.18. Since the function $\bar{F}$ is continuous with respect to the first two variables, if f is continuous and bounded, by Lebesgue dominated convergence theorem $Q_t(f)$ is (bounded) and continuous.

We want to prove the following theorem.

Theorem 7.24. *Let* (μ, σ) *be an autonomous SDE satisfying assumption A and let* X_t *the strong solution to the SDE* (μ, σ) *driven by the Brownian motion* $\bar{B}$ *and with initial condition* $Y \in L^2(\mathcal{F}_0)$ *then, for any* $t \geqslant s$, *we have*

$$\mathbb{E}[G(X_t) | \mathcal{F}_s] = Q_{t-s}(G)(X_s).$$

Remark 7.25. Hereafter, we introduce the concept of strong solution $X^{t_0,Y}$ to the SDE (μ, σ) driven by the Brownian motion $\bar{B}$, starting at the time $t_0 \geqslant 0$ with initial condition $Y \in (L^0(\mathcal{F}_{t_0}))^m$, namely we have that $X^{t_0,Y}$ the process is continuous and it satisfies the first part of Definition 6.1 and we have

$$X_t^{t_0,Y,k} = Y^k + \int_{t_0}^t \mu^k(X_s^{t_0,Y}) \mathrm{d}s + \sum_{j=1}^n \int_{t_0}^t \sigma_j^k(X_s^{t_0,Y,k}) \mathrm{d}B_s^j.$$

Lemma 7.26. *Consider an autonomous SDE* (μ, σ) *satisfying assumption A and let* $t_0 \geqslant 0$, $Y \in (L^0(\mathcal{F}_{t_0}))^m$ *and* $\bar{B}$ *a* n*-dimensional Brownian motion. Then we have that*

$$X_t^{t_0,Y} = \bar{F}(t - t_0, Y, (B_{[t_0,t]} - B_{t_0}))$$

is the unique strong solution to the SDE (μ, σ) *driven by the Brownian motion* $\bar{B}$, *starting at the time* $t_0 \geqslant 0$ *with initial condition* $Y \in (L^0(\mathcal{F}_{t_0}))^m$, *where* $\bar{F}$: $\mathbb{R}_+ \times \mathbb{R}^m \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^m$ *is the map in the thesis of Theorem 7.18.*

Proof. The existence and uniqueness of strong solutions starting at an arbitrary time t_0 can be proved as in Theorem 6.8 and Theorem 6.16.

Consider the stochastic process

$$\tilde{B}_{t-t_0} = \bar{B}_t - \bar{B}_{t_0} = (\bar{B}_t^1 - \bar{B}_{t_0}^1, \ldots, \bar{B}_t^n - \bar{B}_{t_0}^n),$$

defined for $t \geqslant t_0$. The process $\tilde{B}_t$ is a Brownian motion independent of $\mathcal{F}_{t_0}$ and, denoting by $\tilde{\mathcal{F}}_t^{\tilde{B}}$ the natural filtration of $\tilde{B}_t$ we have that

$$\mathcal{F}_t = \sigma\big(\tilde{\mathcal{F}}_{t-t_0}^{\tilde{B}}, \mathcal{F}_{t_0}\big).$$

Let $\tilde{X}_t$ be the strong solution to the SDE (μ, σ) driven by the Brownian motion $\tilde{B}_t$ and with initial condition $Y \in \mathcal{F}_{t_0}$, thus we have that $\tilde{X}_t$ is given by the expression

$$\tilde{X}_t^k = Y^k + \int_0^t \mu(\tilde{X}_s^k) \mathrm{d}s + \sum_{j=1}^n \int_0^t \sigma_j^k(\tilde{X}_s) \mathrm{d}\tilde{B}_s.$$

On the other hand we have that

$$\begin{aligned}\int_0^t \sigma_j^k(\tilde{X}_s)\mathrm{d}\tilde{B}_s^j &= \lim_{|\pi|\to 0}\sum_{t_\ell\in\pi^t}\sigma_j^k(\tilde{X}_{t_{\ell-1}})(\tilde{B}_{t_\ell}^j-\tilde{B}_{t_{\ell-1}}^j)\\ &= \lim_{|\pi|\to 0}\sum_{t_\ell\in\pi^t}\sigma_j^k(\tilde{X}_{t_{\ell-1}})(B_{t_\ell+t_0}^j-B_{t_{\ell-1}+t_0}^j)\\ &= \int_{t_0}^t \sigma_j^k(\tilde{X}_{s-t_0})\mathrm{d}B_s^j.\end{aligned}$$

This means that, if we write $\hat{X}_t := \tilde{X}_{t-t_0}$ (defined for $t\geqslant t_0$), we get that

$$\begin{aligned}\hat{X}_t^k &= Y^k+\int_{t_0}^t \mu(\tilde{X}_{s-t_0}^k)\mathrm{d}s+\sum_{j=1}^n\int_{t_0}^t\sigma_j^k(\tilde{X}_{s-t_0})\mathrm{d}B_s^j\\ &= Y^k+\int_{t_0}^t \mu(\hat{X}_s^k)\mathrm{d}s+\sum_{j=1}^n\int_{t_0}^t\sigma_j^k(\hat{X}_s)\mathrm{d}B_s^j,\end{aligned}$$

and thus $\hat{X}_t^k = X_t^{t_0,Y}$ is the strong solution to the SDE (μ,σ) driven by $\bar{B}$, starting at t_0 with initial condition Y. On the other hand, by Theorem 7.21, we have

$$X_t^{t_0,Y}(\omega)=\hat{X}_t(\omega)=\tilde{X}_{t-t_0}(\omega)=\bar{F}(t-t_0,Y(\omega),\tilde{B}_{[0,t-t_0]})=\bar{F}(t-t_0,Y(\omega),(B_{[t_0,t]}-B_{t_0})),$$

for almost every $\omega\in\Omega$. □

Proof of Theorem 7.24. If X_t is the strong solution to the SDE (μ,σ) driven by the Brownian motion $\bar{B}$, then for any $t\geqslant s$ we have

$$X_t^k - X_s^k = \int_s^t \mu^k(X_s)\mathrm{d}s+\sum_{j=1}^n\int_s^t\sigma_j^k(X_s)\mathrm{d}B_s^j.$$

In other words, for any $t\geqslant s$, $X_t = X_t^{s,X_s}$ where X_t^{s,X_s} is the strong solution to the SDE (μ,σ) starting at $s\geqslant 0$ with initial condition $X_s\in L^2(\mathcal{F}_s)$. On the other hand, by Lemma 7.26, we have

$$X_t^{s,X_s}=\bar{F}(t-s,X_s,\bar{B}_{[s,t]}-\bar{B}_s).$$

Thus we obtain

$$\mathbb{E}[G(X_t)|\mathcal{F}_s]=\mathbb{E}[G(X_t^{s,X_s})|\mathcal{F}_s]=\mathbb{E}[G(\bar{F}(t-s,X_s,\bar{B}_{[s,t]}-\bar{B}_s))|\mathcal{F}_s].$$

Furthermore, since $\bar{B}$ is a Brownian motion and it has independent increments, the $\tilde{B}_{t-s}=\bar{B}_t-\bar{B}_s$ is a Brownian motion independent of $\mathcal{F}_s$ and thus we have

$$\begin{aligned}\mathbb{E}[G(\bar{F}(t-s,X_s,\bar{B}_{[s,t]}-\bar{B}_s))|\mathcal{F}_s](\omega) &= \int_{C^0(\mathbb{R}_+,\mathbb{R}^n)}G(\bar{F}(t-s,X_s(\omega),\tilde{B}_{t-s}(\cdot)))\mathbb{P}_{\tilde{B}_{t-s}}(\mathrm{d}\,\cdot)\\ &= \mathbb{E}_{\tilde{B}}[G(\bar{F}(t-s,X_s(\omega),\tilde{B}_{t-s}(\cdot)))]\end{aligned}$$

where the previous symbols we mean that $\omega\in\Omega$ we fix the value of the random variable $X_s(\omega)$ and we take the expectation with respect to the *independent Brownian motion* $\tilde{B}$. On the other hand, by definition of Q_{t-s}, we have

$$\mathbb{E}_{\tilde{B}}[G(\bar{F}(t-s,X_s(\omega),\tilde{B}_{t-s}(\cdot)))]=Q_{t-s}(G)(X_s).$$ □

Corollary 7.27. *We have that for any $t_1,t_2\geqslant 0$, then, for any $f\in C_b^0(\mathbb{R}^m,\mathbb{R})$, $Q_{t_1+t_2}(f)=Q_{t_2}(Q_{t_1}(f))$.*

Proof. It follows from Theorem 7.24 and the tower property of the expected values. Indeed,

$$\begin{aligned}Q_{t_1+t_2}(f)(x) &= \mathbb{E}[f(X_{t_1+t_2}^x)|\mathcal{F}_0]=\mathbb{E}[\mathbb{E}[f(X_{t_1+t_2}^x)|\mathcal{F}_{t_2}]|\mathcal{F}_0]\\ &= \mathbb{E}[Q_{t_1}(X_{t_2}^x)|\mathcal{F}_0]=Q_{t_2}(Q_{t_1}(f))(X_0^x)=Q_{t_2}(Q_{t_1}(f))(x),\end{aligned}$$

which is the stated property. □

Remark 7.28. Corollary 7.27 proves that the family of maps Q_t is a semigroup. When X_t is a Markov process, the semigroup Q_t is called Markov semigroup associated with the process X_t.

Corollary 7.29. *We have that, for any $t_1, t_2 \geqslant 0$*

$$\bar{F}(t_1 + t_2, x, B_{[0, t_1 + t_2]}) = \bar{F}(t_2, \bar{F}(t_1, x, B_{[0, t_1]}), B_{[t_1, t_1 + t_2]} - B_{t_2}). \tag{7.13}$$

Proof. The statement follows directly from the proof of Theorem 7.24. □

Remark 7.30. Sometimes it is equation (7.13) that is called Markov property of the solution to the SDE (μ, σ). More generally (7.13) show that, defining for every $s \leqslant t \in \mathbb{R}_+$ and $\omega \in \Omega$, the continuous map $\Phi_{(s,t),\omega}: \mathbb{R}^m \rightarrow \mathbb{R}^m$, as

$$\Phi_{(s,t),\omega}(x) := \bar{F}(t - s, x, B_{[s,t]}(\omega) - B_s(\omega)),$$

for any $\omega \in \Omega$, the map $\Phi_{\cdot,\omega}(\cdot)$ is a flow of homeomorphism, namely, for any $s \leqslant t \leqslant u$ we have

$$\Phi_{(s,u),\omega}(x) = \Phi_{(t,u),\omega}(\Phi_{(s,t),\omega}(x)).$$

Theorem 7.31. *Let (μ, σ) be an autonomous SDE satisfying assumption A and let X_t the strong solution to the SDE (μ, σ) driven by the Brownian motion $\bar{B}$ and with initial condition $Y \in L^2(\mathcal{F}_0)$ then X_t is a Markov process.*

Proof. We prove the theorem for functions $G: \mathbb{R}^{mk} \rightarrow \mathbb{R}$ of the form

$$G(X_{t_1}, \ldots, X_{t_k}) = G_1(X_{t_1}) \cdots G_k(X_{t_k}).$$

Since the functions of the previous form are dense (with respect to the point wise convergence) in the set of continuous bounded function, the theorem is proved.

Consider $t \leqslant t_1 \leqslant \cdots \leqslant t_k \in \mathbb{R}_+$, then, by Theorem we have

$$\begin{aligned}
\mathbb{E}[G(X_{t_1}, \ldots, X_{t_k}) | \mathcal{F}_t] &= \mathbb{E}[G_1(X_{t_1}) \cdots G_{k-1}(X_{t_{k-1}}) \mathbb{E}[G_k(X_{t_k}) | \mathcal{F}_{t_{k-1}}] | \mathcal{F}_t] \\
&= \mathbb{E}[G_1(X_{t_1}) \cdots G_{k-1}(X_{t_{k-1}}) Q_{t_k - t_{k-1}}(G)(X_{t_{k-1}}) | \mathcal{F}_t] \\
&= \mathbb{E}[G_1(X_{t_1}) \cdots G_{k-2}(X_{t_{k-2}}) \mathbb{E}[G_{k-1}(X_{t_{k-1}}) Q_{t_k - t_{k-1}}(G)(X_{t_{k-1}}) | \mathcal{F}_{t_{k-2}}] | \mathcal{F}_t] \\
&= \mathbb{E}[G_1(X_{t_1}) \cdots G_{k-2}(X_{t_{k-2}}) Q_{t_{k-1} - t_{k-1}}(G_{k-1} Q_{t_k - t_{k-1}}(G_k))(X_{t_{k-2}}) | \mathcal{F}_t] \\
&= \mathbb{E}[G_1(X_{t_1}) Q_{t_2 - t_1}(G_2 Q_{t_3 - t_2}(\cdots Q_{t_k - t_{k-1}}(G_k) \cdots)(X_{t_1}) \,| \mathcal{F}_t] \\
&= Q_{t_1 - t}(G_1 Q_{t_2 - t_1}(G_2 \cdots Q_{t_k - t_{k-1}}(G_k) \cdots))(X_t).
\end{aligned}$$

In other words $\mathbb{E}[G(X_{t_1}, \ldots, X_{t_k}) | \mathcal{F}_t]$ is equal to a function of only X_t, i.e.

$$\begin{aligned}
\mathbb{E}[G(X_{t_1}, \ldots, X_{t_k}) | X_t] &= \mathbb{E}[\mathbb{E}[G(X_{t_1}, \ldots, X_{t_k}) | \mathcal{F}_t] | X_t] \\
&= \mathbb{E}[Q_{t_1 - t}(G_1 Q_{t_2 - t_1}(G_2 \cdots Q_{t_k - t_{k-1}}(G_k) \cdots))(X_t) \,|\, X_t] \\
&= Q_{t_1 - t}(G_1 Q_{t_2 - t_1}(G_2 \cdots Q_{t_k - t_{k-1}}(G_k) \cdots))(X_t).
\end{aligned}$$

This proves that X_t is a Markov process. □

Chapter 8

SDEs and evolution PDEs

8.1 Kolmogorov (backward) equation

We recall the definition of the operator

$$\mathcal{L}_t(f)(t,x) := \sum_{k=1}^{m} \mu^k(t,x)\partial_{x^k} f(t,x) + \frac{1}{2}\sum_{k,k'=1}^{m}\left(\sum_{j=1}^{n} \sigma_j^k(t,x)\sigma_j^{k'}(t,x)\right)\partial_{x^k x^{k'}} f(t,x).$$

Hereafter we denote by $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})$ the set of functions $u: \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ which are differentiable one time with respect to the first variable (i.e. $t \in \mathbb{R}_+$) and two times differentiable with respect to the second set of variables (i.e. $x \in \mathbb{R}^m$), and all the derivatives of u mentioned before are continuous. We will use also the notation $C^{1,2}([0,\tau] \times \mathbb{R}^m, \mathbb{R})$ for the set of functions defined only on the compact set $[0,\tau]$ which are differentiable one time with respect to the first variable and two times differentiable with respect to the second set of variables.

Theorem 8.1. *Let (μ,σ) be a SDE satisfying assumption A, and consider a function $u \in C^{1,2}([0,\tau] \times \mathbb{R}^m, \mathbb{R})$ which is a (classical) solution to the PDE*

$$\partial_t u(t,x) + \mathcal{L}_t u(t,x) = 0, \quad u(\tau,x) = f(x) \tag{8.1}$$

(where $f(x) \in C^2(\mathbb{R}^m, \mathbb{R})$) and such that u grows at most polynomial at infinity, i.e. there is $N \in \mathbb{N}$ and $R > 0$ such that, for every $t \in [0,\tau], x \in \mathbb{R}^m$ we have

$$|u(t,x)| \leqslant R(1 + |x|^N).$$

Then we have

$$\mathbb{E}[f(X_\tau^{t,x})] = \mathbb{E}[u(\tau, X_\tau^{t,x})] = u(t,x), \tag{8.2}$$

where $X_s^{t,x}$ is a the strong solution to the SDE (μ,σ) starting at time $t \in \mathbb{R}_+$ and with initial condition $X_x^{t,x} = x \in \mathbb{R}^m$.

Proof. The theorem is a simple application of Ito formula. Indeed, by Lemma 7.10, we have

$$\begin{aligned} \mathrm{d}u(s, X_s^{t,x}) &= (\partial_t u(s, X_s^{t,x}) + \mathcal{L}_s(u)(s, X_s^{t,x}))\mathrm{d}s + \sum_{j=1}^{n}\left(\sum_{k=1}^{m} \sigma_j^k(s, X_s^{t,x})\partial_{x^k} u(s, X_s^{t,x})\right)\mathrm{d}B_s^j \\ &= \sum_{j=1}^{n}\left(\sum_{k=1}^{m} \sigma_j^k(s, X_s^{t,x})\partial_{x^k} u(s, X_s^{t,x})\right)\mathrm{d}B_s^j, \end{aligned}$$

where we used that u satisfies equation (8.1). The previous equality proves that the process $u(s, X_s^{t,x})$ (defined for $s \geqslant t$) is a local martingale.

We now prove that $u(s, X_s^{t,x})$ is a real martingale. Since the initial condition of the strong solution $X_s^{t,x}$ to the SDE (μ,σ) is deterministic by Theorem 7.14, we have

$$\mathbb{E}[|X_s^{t,x}|^p] \leqslant C_{K,p}\, e^{\lambda_{K,p}(s-t)}(1 + |x|^p),$$

for every $s>0$ and $p\geqslant 4$ and suitable constant $C_{K,p},\lambda_{K,p}\geqslant 0$. Thus we have

$$\begin{aligned}\mathbb{E}[|u(s,X_s^{t,x})|^2] &\leqslant \mathbb{E}[(1+|X_s^{t,x}|)^{2N}]\leqslant 2^{2N}(1+\mathbb{E}[|X_s^{t,x}|^{2N}])\\ &\leqslant 2^{2N}(1+C_{K,2N}e^{\lambda_{K,2N}(s-t)}(1+|x|^{2N}))<+\infty.\end{aligned}$$

This implies that the process $u(s,X_s^{t,x})$ is a continuous local martingale bounded in L^2, which means that $u(s,X_s^{t,x})$ is a $\mathcal{M}_c^2$ martingale and so a real martingale. Thus by definition of martingale

$$\mathbb{E}[f(X_\tau^{t,x})]=\mathbb{E}[u(\tau,X_\tau^{t,x})]=\mathbb{E}[\mathbb{E}[u(\tau,X_\tau^{t,x})|\mathcal{F}_t]]=\mathbb{E}[u(t,X_t^{t,x})]=\mathbb{E}[u(t,x)]=u(t,x),$$

which concludes the proof. □

Remark 8.2. Equation (8.1) is usually called the *Kolmogorov (backward) equation associated with the SDE* (μ,σ).

For autonomous SDE, i.e. in the case where μ,σ do not depend on t and so the operator

$$\mathcal{L}_t:=\mathcal{L}=\sum_{k=1}^m \mu^k(x)\partial_{x^k}+\frac{1}{2}\sum_{k,k'=1}^m\left(\sum_{j=1}^n\sigma_j^k(x)\sigma_j^{k'}(x)\right)\partial_{x^kx^{k'}} \tag{8.3}$$

do not depend on t too, we can give to Theorem 8.1 the following formulation.

Theorem 8.3. *Let (μ,σ) be an autonomous SDE satisfying assumption A, and consider a function $v\in C^{1,2}([0,\tau]\times\mathbb{R}^m,\mathbb{R})$ which is a (classical) solution to the PDE*

$$\partial_t v(t,x)=\mathcal{L}v(t,x),\quad v(0,x)=f(x) \tag{8.4}$$

(where $f(x)\in C^2(\mathbb{R}^m,\mathbb{R})$) and such that v grows at most polynomial at infinity, i.e. there is $N\in\mathbb{N}$ and $R>0$ such that, for every $t\in[0,\tau],x\in\mathbb{R}^m$, we have

$$|v(t,x)|\leqslant R(1+|x|^N).$$

Then we have

$$\mathbb{E}[f(X_t^x)]=\mathbb{E}[v(0,X_t^x)]=v(t,x), \tag{8.5}$$

where X_t^x is the strong solution to the SDE (μ,σ) starting at time 0 and with initial condition $X_0^x=x\in\mathbb{R}^m$.

Remark 8.4. Using the map $\bar{F}:\mathbb{R}_+\times\mathbb{R}^m\times C^0(\mathbb{R}_+,\mathbb{R}^n)\to\mathbb{R}^m$ defined in Theorem 7.18, equation (8.5) can be written in the following way

$$v(x,t)=\mathbb{E}[f(\bar{F}(t,x,\bar{B}_{[0,t]}))]=Q_t(f)(x).$$

Proof of Theorem 8.3. Fix $\tau>0$ and consider the function

$$u(t,x)=v(\tau-t,x).$$

Then the function u solves the Kolmogorov backward equation associated with (μ,σ); indeed

$$\partial_t u(t,x)=-\partial_t v(\tau-t,x)=-\mathcal{L}v(\tau-t,x)=-\mathcal{L}u(t,x).$$

By Theorem 8.1, this means that, for any $0\leqslant t\leqslant\tau$, we have

$$v(\tau-t,x)=u(t,x)=\mathbb{E}[u(\tau,X_\tau^{t,x})]=\mathbb{E}[v(0,X_\tau^{t,x})]=\mathbb{E}[f(X_\tau^{t,x})].$$

On the other hand by Lemma 7.26, we have

$$v(\tau-t,x)=\mathbb{E}[f(X_\tau^{t,x})]=\mathbb{E}[f(\bar{F}(\tau-t,x,\bar{B}_{[0,\tau-t]}))]$$

which, by Remark 8.4, is equivalent to the thesis. □

We can prove a sort of reverse of Theorem 8.3.

Proposition 8.5. *Consider $f\in C^2(\mathbb{R}^m,\mathbb{R})$ and suppose that (μ,σ) satisfies assumption A. If the function $u(t,x)=Q_t(f)(x)$ (as a function of $(t,x)\in\mathbb{R}_+\times\mathbb{R}^m$) is such that for any $t\in\mathbb{R}_+$ we have $u(t,\cdot)\in C^2(\mathbb{R}^m,\mathbb{R})$ and for any $x\in\mathbb{R}^m$ we have that $u(\cdot,x)$ is differentiable with respect to time, and the function $u(t,x)$ and its first and second derivatives have at most polynomial growth at infinity, then we have*

$$\partial_t u=\mathcal{L}(u)(t,x).$$

Proof. By the semigroup property of Q_t we have that for any $\Delta t\geqslant 0$

$$u(t,+\Delta t,x)=Q_{t+\Delta t}f(x)=Q_{\Delta t}(Q_t f(x))=Q_{\Delta t}(u(t,x)).$$

Furthermore, by definition of $Q_{\Delta t}$, we have

$$Q_{\Delta t}(u(t,x))=\mathbb{E}[u(t,X^x_{\Delta t})].$$

Since $u\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^m,\mathbb{R})$ we can apply Ito formula to $u(t,X^x_{\Delta t})$, getting

$$\mathbb{E}[u(t,X^x_{\Delta t})]=u(t,x)+\mathbb{E}\left[\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\right]+\mathbb{E}\left[\sum_{k=1}^m\sum_{j=1}^n\int_0^{\Delta t}\sigma^k_j(X^x_s)\partial_{x^k}u(t,X^x_s)\mathrm{d}B^j_s\right].$$

Since $\partial_{x^k}u(t,x)$ has at most linear growth at infinity then

$$\mathbb{E}\left[\int_0^{\Delta t}|\sigma^k_j(X^x_s)\partial_{x^k}u(t,X^x_s)|^2\mathrm{d}s\right]<+\infty$$

and so $\int_0^{\Delta t}\sigma^k_j(X^x_s)\partial_{x^k}u(t,X^x_s)\mathrm{d}B^j_s$ is a real martingale. This implies that

$$\frac{\mathbb{E}[u(t,X^x_{\Delta t})]-u(t,x)}{\Delta t}=\mathbb{E}\left[\frac{1}{\Delta t}\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\right].$$

We have that, for any $p\geqslant 1$ and $\Delta t\leqslant 1$,

$$\mathbb{E}[|u(t,X^x_{\Delta t})|^p]<C_p,$$

and

$$\mathbb{E}\left[\left|\frac{1}{\Delta t}\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\right|^p\right]\leqslant^{\text{Jensen}}\frac{1}{\Delta t}\int_0^{\Delta t}\mathbb{E}[|\mathcal{L}(u)(t,X^x_s)|^p]<C_p,$$

for some constant $C_p>0$ (dependent on p but not on Δt). Thus the random variables $\{u(t,X^x_{\Delta t})\}_{\Delta t\leqslant 1}$ and $\left\{\frac{1}{\Delta t}\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\right\}_{\Delta t}$ are uniformly integrable.

Finally, since $X^x_t\to x$ as $t\to 0$ and $\mathcal{L}(u)$ is continuous (being $u\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^m,\mathbb{R})$), we have

$$\frac{1}{\Delta t}\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\to\mathcal{L}(u)(t,x),\quad \Delta t\to 0,$$

almost surely. By the uniform integrability of $\left\{\frac{1}{\Delta t}\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\right\}_{\Delta t\leqslant 1}$, this implies that

$$\lim_{\Delta t\to 0}\mathbb{E}\left[\frac{1}{\Delta t}\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\right]=\mathcal{L}(u)(t,x).$$

Thus we have

$$\begin{aligned}\partial_t u(t,x) &= \lim_{\Delta t\to 0^+}\frac{(u(t+\Delta t,x)-u(t,x))}{\Delta t}\\ &= \lim_{\Delta t\to 0^+}\frac{(Q_{\Delta t}u(t,x)-u(t,x))}{\Delta t}\\ &= \lim_{\Delta t\to 0^+}\frac{(Q_{\Delta t}u(t,x)-u(t,x))}{\Delta t}\\ &= \lim_{\Delta t\to 0^+}\frac{\mathbb{E}[u(t,X^x_{\Delta t})]-u(t,x)}{\Delta t}\\ &= \lim_{\Delta t\to 0^+}\mathbb{E}\left[\frac{1}{\Delta t}\int_0^{\Delta t}\mathcal{L}(u)(t,X^x_s)\mathrm{d}s\right]=\mathcal{L}(u)(t,x).\end{aligned}$$

□

8.1.1 Feynman-Kac formula

The probabilistic representations (8.2) and (8.5), of the solutions to equations (8.1) and (8.4) respectively, can be extended to a more general linear parabolic equations.

Definition 8.6. *Consider a continuous function $q:\mathbb{R}^m\to\mathbb{R}$ bounded from below, an autonomous SDE (μ,σ) satisfying assumption A (and the related operator $\mathcal{L}$ defined in (8.3)), a $C^{1,2}(\mathbb{R}_+\times\mathbb{R}^m,\mathbb{R})$ function $g:\mathbb{R}_+\times\mathbb{R}^m\to\mathbb{R}$, and a $C^2(\mathbb{R}^m,\mathbb{R})$ function $f:\mathbb{R}^m\to\mathbb{R}$ growing at most polynomially at infinity. Then a function $v\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^m,\mathbb{R})$ growing at most polynomially at infinity is a classical solution to the parabolic equation $(q,\mathcal{L},g)$ with initial condition f if*

$$\partial_t v(t,x)=\mathcal{L}v(t,x)-q(x)v(t,x)-g(t,x),\quad v(0,x)=f(x), \tag{8.6}$$

for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^m$.

Theorem 8.7. *Under the hypotheses and notation of Definition 8.6, if v is a solution to equation (8.6) growing at most polynomially at infinity then we have that*

$$v(t,x)=\mathbb{E}\left[f(X_t^x)e^{-\int_0^t q(X_s^x)\mathrm{d}s}+\int_0^t g(s,X_s^x)e^{-\int_0^s q(X_\ell^x)\mathrm{d}\ell}\mathrm{d}s\right], \tag{8.7}$$

where X_t^x is the strong solution to the SDE (μ,σ) such that $X_0^x=x\in\mathbb{R}^m$.

Proof. Fix $\tau>0$ and consider the process (defined in the set $[0,\tau]$)

$$R_t=v(\tau-t,X_t^x)e^{-\int_0^t q(X_s^x)\mathrm{d}s}+\int_0^t g(s,X_s^x)e^{-\int_0^s q(X_\ell)\mathrm{d}\ell}\mathrm{d}s.$$

We have that

$$\begin{aligned}
\mathrm{d}R_t &= \left(-\partial_t v(\tau-t,X_t^x)e^{-\int_0^t q(X_s^x)\mathrm{d}s}-q(X_t^x)v(\tau-t,X_t^x)e^{-\int_0^t q(X_s^x)\mathrm{d}s}\right)\mathrm{d}t\\
&\quad+\left(\mathcal{L}v(t-\tau,X_t^x)e^{-\int_0^t q(X_s^x)\mathrm{d}s}+g(t,X_s^x)e^{-\int_0^t q(X_s)\mathrm{d}s}\right)\mathrm{d}t\\
&\quad+\sum_{j=1}^n\left(\sum_{k=1}^m\sigma_j^k(X_t^x)\partial_{x^k}v(t-\tau,X_t^x)\right)\mathrm{d}B_t^j\\
&= \sum_{j=1}^n\left(\sum_{k=1}^m\sigma_j^k(X_t^x)\partial_{x^k}v(t-\tau,X_t^x)\right)\mathrm{d}B_t^j,
\end{aligned}$$

thus R_t is a local martingale. Using the fact that $\mathbb{E}[|X_t^x|^p]<+\infty$ for any $p\geqslant 1$, in a way similar to what was done in the proof of Theorem 8.1, it is possible to prove that R_t is a real martingale. Thus, we get

$$\begin{aligned}
&\mathbb{E}\left[f(X_\tau^x)e^{-\int_0^\tau q(X_s^x)\mathrm{d}s}+\int_0^\tau g(s,X_s^x)e^{-\int_0^s q(X_\ell)\mathrm{d}\ell}\mathrm{d}s\right]\\
&= \mathbb{E}\left[v(\tau-\tau,X_t^x)e^{-\int_0^\tau q(X_s^x)\mathrm{d}s}+\int_0^\tau g(s,X_s^x)e^{-\int_0^s q(X_\ell)\mathrm{d}\ell}\mathrm{d}s\right]\\
&= \mathbb{E}[R_\tau]=\mathbb{E}[\mathbb{E}[R_\tau|\mathcal{F}_0]]=\mathbb{E}[R_0]=\mathbb{E}[v(\tau,X_0^x)]=v(\tau,x),
\end{aligned}$$

which gives the result. □

8.2 Existence of solution to Kolmogorov PDE: Ornstein-Uhlenbeck case

Let use consider the Ornstein-Uhlenbeck equation, namely

$$X_t^x=x+\int_0^t\alpha X_s^x\mathrm{d}s+B_t, \tag{8.8}$$

where $\alpha \in \mathbb{R}$. In Section 6.1.1.2 the explicit solution to the equation (6.6) is given by the expression

$$X_t^x = e^{\alpha t}x + \int_0^t e^{\alpha(t-s)}\mathrm{d}B_s.$$

Let $f \in C^2(\mathbb{R}, \mathbb{R})$ be a function such that f and its first and second derivatives grow at most polynomially at infinity. Define the function

$$v(t,x) = \mathbb{E}[f(X_t^x)] = \mathbb{E}\left[f\left(e^{\alpha t}x + \int_0^t e^{\alpha(t-s)}\mathrm{d}B_s\right)\right].$$

Proposition 8.8. *The function v is in $C^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and it satisfies the equation*

$$\partial_t v(t,x) = \frac{1}{2}\partial_{xx}^2 v(t,x) + \alpha x \partial_x v(t,x), \quad v(0,x) = f(x).$$

Proof. The proof is given for the general case below. □

8.3 Regularity of SDEs with additive noise

Definition 8.9. *Let (μ, σ) be a SDE we say that the coefficients (or the SDE) μ, σ satisfy assumption B, if (μ, σ) is an autonomous SDE and, for every $k = 1, \ldots, m$ and $j = 1, \ldots, n$, we have that $\mu^k, \sigma_j^k \in C^2(\mathbb{R}^m, \mathbb{R})$ and there is $K > 0$ such that, for any $h, h' = 1, \ldots, m$, we have*

$$|\partial_{x^h}\mu^k(x)|, |\partial_{x^h}\sigma_j^k(x)|, |\partial_{x^h x^{h'}}\mu^k(x)|, |\partial_{x^h x^{h'}}\sigma_j^k(x)| \leqslant K, \quad x \in \mathbb{R}^m.$$

Definition 8.10. *We say that a SDE (μ, σ) is an additive noise SDE if $\sigma = \mathrm{cost}$ does not depend on $t, x \in \mathbb{R}_+ \times \mathbb{R}^m$.*

Let X_t^x be a solution to an additive noise SDE, i.e.

$$X_t^{x,k} = x^k + \int_0^t \mu^k(X_s^x)\mathrm{d}s + \sum_{j=1}^n \sigma_j^k B_t^j$$

(where in the last term we use the fact that

$$\int_0^t \sigma_j^k \mathrm{d}B_s^j = \sigma_j^k B_t^j$$

being $\sigma_j^k \in \mathbb{R}$).

Theorem 8.11. *Let (μ, σ) be a SDE with additive noise satisfying assumption B, then, for every $t \in \mathbb{R}_+$ and $\omega \in \Omega$, the map $x \mapsto X_t^x(\omega)$ is $C^2(\mathbb{R}^m, \mathbb{R}^m)$. Furthermore, if we define, for every h, h', $k = 1, \ldots, m$, the processes*

$$\xi_{h,t}^{x,k}(\omega) = \partial_{x^h} X_t^x(\omega), \quad \chi_{h,h',t}^{x,k}(\omega) = \partial_{x^h x^{h'}} X_t^x(\omega),$$

then they satisfy the (random) ODEs

$$\frac{\mathrm{d}\xi_{h,t}^{x,k}(\omega)}{\mathrm{d}t} = \sum_{\ell=1}^m \partial_{x^\ell}\mu^k(X_t^x(\omega))\xi_{h,t}^{x,\ell}(\omega) = \sum_{\ell=1}^m A_\ell^k(t,x,\omega)\xi_{h,t}^{x,\ell}(\omega), \tag{8.9}$$

$$\begin{aligned}\frac{\mathrm{d}\chi_{h,h',t}^{x,k}(\omega)}{\mathrm{d}t} &= \sum_{\ell=1}^m \partial_{x^\ell}\mu^k(X_t^x(\omega))\chi_{h,h',t}^{x,\ell}(\omega) + \sum_{\ell,\ell'=1}^m \partial_{x^\ell x^{\ell'}}\mu^k(X_t^x(\omega))\xi_{h,t}^{x,\ell}(\omega)\xi_{h',t}^{x,\ell'}(\omega) \\ &= \sum_{\ell=1}^m A_\ell^k(t,x,\omega)\chi_{h,h',t}^{x,k}(\omega) + \sum_{\ell,\ell'=1}^m B_{\ell,\ell'}^k(t,x,\omega)\xi_{h,t}^{x,\ell}(\omega)\xi_{h',t}^{x,\ell'}(\omega).\end{aligned} \tag{8.10}$$

Remark 8.12. In order to have a simpler notation, we denote by $A(t,x,\omega):=(A^k_\ell(t,x,\omega))_{\ell,k=1,\ldots,m}$ the matrix in $\mathrm{Mat}(m,m)$ associated with A^k_ℓ and by

$$B(t,x,\omega)[a,b]=\left(\sum_{\ell,\ell'=1}^m B^k_{\ell,\ell'}(t,x,\omega)a^\ell b^{\ell'}\right)$$

the quadratic for associated with $B^k_{\ell,\ell'}(t,x,\omega)$. Adopting this notation, we have that equations (8.11) and equations (8.9) become

$$\frac{\mathrm{d}\xi^x_{h,t}}{\mathrm{d}t}=A(t,x,\omega)\cdot\xi^x_{h,t}\,, \tag{8.11}$$

$$\frac{\mathrm{d}\chi^x_{h,h',t}}{\mathrm{d}t}=A(t,x,\omega)\cdot\chi^x_{h,h',t}+B(t,x,\omega)[\xi^x_{h,t},\xi^x_{h',t}]. \tag{8.12}$$

Proof. We provide a complete proof only for $\xi^x_{h,t}$. For the second derivatives $\chi^x_{h,h',t}$ the proof is similar.

Let $\{e_k\}_{k=1,\ldots,m}\subset\mathbb{R}^m$ be the standard basis of $\mathbb{R}^m$ and consider

$$\Delta^{k,\lambda}X^x_t=\frac{X^{x+\lambda e_k}_t-X^x_t}{\lambda}.$$

We have that

$$\lim_{\lambda\to 0}\Delta^{k,\lambda}X^x_t(\omega)=\xi^x_{k,t}(\omega)$$

if the limit exists. We have that the process $\Delta^{k,\lambda}X^x_t$ solves the following SDE

$$\begin{aligned}
\mathrm{d}\Delta^{k,\lambda}X^{x,h}_t &= \frac{1}{\lambda}(\mu^h(X^{x+\lambda e_k}_t)-\mu^h(X^x_t))\mathrm{d}t+\frac{1}{\lambda}\sum_{j=1}^n\sigma^h_j\mathrm{d}B^j_t-\frac{1}{\lambda}\sum_{j=1}^n\sigma^h_j\mathrm{d}B^j_t\\
&= \frac{1}{\lambda}(\mu^h(X^{x+\lambda e_k}_t)-\mu^h(X^x_t))\mathrm{d}t\\
&= \left(\sum_{\ell=1}^m\int_0^1\partial_{x^\ell}\mu^h(\tau X^{x+\lambda e_k}_t+(1-\tau)X^x_t)\frac{(X^{x+\lambda e_k,\ell}_t-X^{x+\lambda e_k,\ell}_t)}{\lambda}\mathrm{d}\tau\right)\mathrm{d}t\\
&= \sum_{\ell=1}^m\left(\int_0^1\partial_{x^\ell}\mu^h(\tau X^{x+\lambda e_k}_t+(1-\tau)X^x_t)\mathrm{d}\tau\right)\Delta^{k,\lambda}X^{x,\ell}_t\mathrm{d}t\\
&= \sum_{\ell=1}^m\tilde{A}^{h,\lambda}_\ell(t,x,\omega)\Delta^{\ell,\lambda}X^{x,\ell}_t\mathrm{d}t,
\end{aligned}$$

with the initial condition

$$\Delta^{k,\lambda}X^{x,h}_0=\delta^{k,h}.$$

The solution to the previous (random) ODE can be explicitly computed and it has the following form

$$\begin{aligned}
&\Delta^{k,\lambda}X^x_t\\
=\;&\Delta^{k,\lambda}X^x_0+\int_0^t\tilde{A}^\lambda(s_1,x,\omega)\cdot\Delta^{k,\lambda}X^x_0\mathrm{d}s_1+\int_0^t\int_0^{s_1}\tilde{A}^\lambda(s_1,x,\omega)\cdot\tilde{A}^\lambda(s_2,x,\omega)\cdot\Delta^{k,\lambda}X^x_0\mathrm{d}s_1\mathrm{d}s_2\\
&+\ldots+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{N-1}}\tilde{A}^\lambda(s_1,x,\omega)\cdot\tilde{A}^\lambda(s_2,x,\omega)\cdots\tilde{A}^\lambda(s_N,x,\omega)\Delta^{k,\lambda}X^x_0\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_N+\ldots
\end{aligned} \tag{8.13}$$

Since, by assumption B,

$$\sup_{x\in\mathbb{R}^m,s\in\mathbb{R}_+}\|\tilde{A}^\lambda(s,x,\omega)\|_{\mathrm{Mat}(n,n)}\leqslant C,$$

and so we have that

$$\sup_{\substack{x\in\mathbb{R}^m\\ t\in[0,\tau]}}\left\|\int_0^t\int_0^{s_1}\cdots\int_0^{s_{N-1}}\tilde{A}^\lambda(s_1,x,\omega)\cdot\tilde{A}^\lambda(s_2,x,\omega)\cdots\tilde{A}^\lambda(s_N,x,\omega)\Delta^{k,\lambda}X^x_0\,\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_N\right\|\leqslant\frac{\tau^NC^N}{N!}, \tag{8.14}$$

thus the series (8.13) converges uniformly in $t, x \in \mathbb{R}_+ \times \mathbb{R}^m$.

Furthermore, for any $K \subset \mathbb{R}^m$ compact set, we have

$$\begin{aligned}
&\sup_{t\in[0,\tau],x\in K} \|\tilde{A}^\lambda(t,x,\omega) - A(t,x,\omega)\| \\
\leqslant\; & m^2 \sup_{t\in[0,\tau],x\in K,\ell,h=1,\ldots,m} \left\| \int_0^1 \partial_{x^\ell}\mu^h(rX_t^{x+\lambda e_k}(\omega) + (1-r)X_t^x(\omega))\mathrm{d}r - \int_0^1 \partial_{x^\ell}\mu^h(X_t^x(\omega))\mathrm{d}r \right\| \\
\leqslant\; & m^2 \sup_{t\in[0,\tau],r\in[0,1],x\in K,\ell,h=1,\ldots,m} |\partial_{x^\ell}\mu^h(rX_t^{x+\lambda e_k}(\omega) + (1-r)X_t^x(\omega)) - \partial_{x^\ell}\mu^h(X_t^x(\omega))|.
\end{aligned}$$

Since the function

$$(t,r,x,\lambda) \mapsto \partial_{x^\ell}\mu^h(rX_t^{x+\lambda e_k}(\omega) + (1-r)X_t^x(\omega)) - \partial_{x^\ell}\mu^h(X_t^x(\omega))$$

is continuous (being the function $(t,y) \mapsto X_t^y(\omega)$ continuous), and thus it is uniformly continuous when (t,r,x,λ) are in the compact set $[0,\tau]\times[0,1]\times K\times[0,1]$, and since

$$\lim_{\lambda\to 0} (\partial_{x^\ell}\mu^h(rX_t^{x+\lambda e_k}(\omega) + (1-r)X_t^x(\omega)) - \partial_{x^\ell}\mu^h(X_t^x(\omega))) = 0,$$

by the uniform continuity we get

$$\begin{aligned}
&\lim_{\lambda\to 0}\sup_{t\in[0,\tau],x\in K} \|\tilde{A}^\lambda(t,x,\omega) - A(t,x,\omega)\| \\
\leqslant\; & m^2 \lim_{\lambda\to 0} \sup_{\substack{t\in[0,\tau],r\in[0,1],\\ x\in K,\ell,h=1,\ldots,m}} |\partial_{x^\ell}\mu^h(rX_t^{x+\lambda e_k}(\omega) + (1-r)X_t^x(\omega)) - \partial_{x^\ell}\mu^h(X_t^x(\omega))| \\
=\; & 0.
\end{aligned}$$

This implies also that, for any compact set $\mathcal{K} \subset \mathbb{R}^m$, $\tau > 0$ and $N \in \mathbb{N}$,

$$\begin{aligned}
\lim_{\lambda\to 0}\sup_{t\in[0,\tau],x\in K} \Bigg\| &\int_0^t\int_0^{s_1}\cdots\int_0^{s_{N-1}} \tilde{A}^\lambda(s_1,x,\omega)\cdot\tilde{A}^\lambda(s_2,x,\omega)\cdots\tilde{A}^\lambda(s_N,x,\omega)\Delta^{k,\lambda}X_0^x\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_N \\
&-\int_0^t\int_0^{s_1}\cdots\int_0^{s_{N-1}} A(s_1,x,\omega)\cdot A(s_2,x,\omega)\cdots A(s_N,x,\omega)\Delta^{k,\lambda}X_0^x\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_N \Bigg\| = 0.
\end{aligned}$$

Using the uniform bound (8.14) we obtain that

$$\begin{aligned}
\lim_{\lambda\to 0}\Delta^{k,\lambda}X_t^x = \;& f_k + \int_0^t A(s_1,x,\omega)\cdot e_k\mathrm{d}s_1 + \cdots + \\
&+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{N-1}} A(s_1,x,\omega)\cdot A(s_2,x,\omega)\cdots A(s_N,x,\omega)e_k\,\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_N + \ldots
\end{aligned} \tag{8.15}$$

where

$$f_k = (f_k^1,\ldots,f_k^m) \in \mathbb{R}^m, \quad f_k^j = \delta_k^j$$

(and δ_k^j is the Kronecker delta). On the other hand, the left hand side of equation (8.15) is equal to

$$\lim_{\lambda\to 0}\Delta^{k,\lambda}X_t^x = \xi_{k,t}^x$$

and the right hand side of equation (8.15) is the explicit expression of the solution to the ODE

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = A(t,x,\omega)\cdot z(t), \quad z(0) = f_k. \tag{8.16}$$

Thus, $\xi_{k,t}^x$ exists finite and it is solution to equation (8.16) (which is exactly equation (8.11)). □

Corollary 8.13. *Suppose that (μ,σ) satisfies the hypotheses of Theorem 8.11, then there is $C > 0$ such that, for any $t \in \mathbb{R}_+$, $x \in \mathbb{R}^m$, $\omega \in \Omega$, $k, \ell, \ell' = 1,\ldots,m$, we have*

$$|\xi_{\ell,t}^{x,k}| \leqslant Ce^{Ct}, \quad |\chi_{\ell,t}^{x,k}| \leqslant Ce^{Ct}.$$

Proof. The result is easy consequence of inequality (8.14). □

8.4 Existence of solutions to Kolmogorov equation: additive noise case

We recall the definition of the semigroup Q_t associate with an (autonomous) SDE (μ,σ):

$$Q_t(f)(x) := \mathbb{E}[f(X_t^x)] = \mathbb{E}[f(\bar{F}(t,x,B_{[0,t]}))].$$

Theorem 8.14. *Consider $f \in C^2(\mathbb{R}^m,\mathbb{R})$ and suppose that f and its first and second derivatives have at most polynomial growth at infinity. Suppose that (μ,σ) is an autonomous additive noise SDE satisfying assumption B. Then the Kolmogorov equation*

$$\partial_t u = \mathcal{L}(u)(t,x), \quad u(0,x) = f(x), \tag{8.17}$$

admits a unique classical solution $u \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})$ such that u has at most polynomial growth at infinity and we have

$$u(t,x) = Q_t(f)(x) = \mathbb{E}[f(\bar{F}(t,x,B_{[0,t]}))] = \mathbb{E}[f(X_t^x)].$$

Lemma 8.15. *Let $g: \mathbb{R}^k \times \Omega \to \mathbb{R}$ be a function which, for any $\omega \in \Omega$, is continuous in $x \in \mathbb{R}^k$ and suppose that there is $p > 1$, for any compact $\mathcal{K} \subset \mathbb{R}^k$, we have*

$$\sup_{x \in \mathcal{K}} \mathbb{E}[|g(x,\cdot)|^p] < C_{\mathcal{K}} \tag{8.18}$$

for some constant $C_{\mathcal{K}}$. Then the function $G(x) := \mathbb{E}[g(x,\omega)]$ is continuous in $x \in \mathbb{R}^k$. If furthermore $g(\cdot,\omega) \in C^1(\mathbb{R}^m,\mathbb{R})$ and we have

$$\left(\sup_{x \in \mathcal{K}, \ell = 1,\ldots,k} \mathbb{E}[|\partial_{x^\ell} g(x,\cdot)|^p] \right) \leqslant C_{\mathcal{K}}, \tag{8.19}$$

the function $G(x)$ is $C^1(\mathbb{R}^k,\mathbb{R})$ and we have

$$\partial_{x^\ell} G(x) = \mathbb{E}[\partial_{x^\ell} g(x,\cdot)], \quad x \in \mathbb{R}^k, \ell = 1,\ldots,k.$$

Proof. Consider $x \in \mathbb{R}^k$ and let $B_1(x) \subset \mathbb{R}^k$ be the (closed) ball of radius 1 and center $x \in \mathbb{R}^k$. Then, by condition (8.18), the family of random variables $\{g(y,\omega)\}_{y \in B_1(x)}$ is uniformly integrable. Furthermore since, for any $\omega \in \Omega$, $g(\cdot,\omega)$ is continuous we have

$$\lim_{y \to x, y \in B_1(x)} g(y,\omega) = g(x,\omega). \tag{8.20}$$

Since the family $\{g(y,\omega)\}_{y \in B_1(x)}$ is uniformly integrable, the limit (8.20) is not only pointwise (in $\omega \in \Omega$) but in $L^1(\Omega)$. Thus we have

$$\lim_{y \to x, y \in B_1(x)} G(y) = \lim_{y \to x, y \in B_1(x)} \mathbb{E}[g(y,\omega)] = \mathbb{E}\left[\lim_{y \to x, y \in B_1(x)} g(y,\omega) \right] = \mathbb{E}[g(x,\omega)] = G(x)$$

where we can exchange the limit with the expectation since the convergence (8.20) is in $L^1(\Omega)$. Let $\{e_\ell\}_{\ell=1,\ldots,k}$ be the standard basis of $\mathbb{R}^k$. By the fundamental theorem of calculus, we have that

$$g(x + \lambda e_\ell, \omega) - g(x,\omega) = \lambda \int_0^1 \partial_{x^\ell} g(x + \sigma e_\ell, \omega) \mathrm{d}\sigma.$$

We have that the family of random variables

$$\left\{ \frac{g(x + \lambda e_\ell, \omega) - g(x,\omega)}{\lambda} \right\}_{0 < |\lambda| \leqslant 1} \tag{8.21}$$

is uniformly integrable. Indeed

$$\begin{aligned} \mathbb{E}\left[\left| \frac{g(x + \lambda e_\ell, \omega) - g(x,\omega)}{\lambda} \right|^p \right] &= \mathbb{E}\left[\left| \int_0^1 \partial_{x^\ell} g(x + \sigma e_\ell, \omega) \mathrm{d}\sigma \right| \right] \leqslant \int_0^1 \mathbb{E}[|\partial_{x^\ell} g(x + \sigma e_\ell, \omega)|^p] \mathrm{d}\sigma \\ &\leqslant \sup_{y \in B_1(x)} \mathbb{E}[|\partial_{x^\ell} g(x + \sigma e_\ell, \omega)|^p] \leqslant C_{B_1(x)}, \end{aligned}$$

where $B_1(x) \subset \mathbb{R}^k$ is the ball of radius 1 and center $x \in \mathbb{R}^k$. Since $g(\cdot,\omega) \in C^1(\mathbb{R}^k,\mathbb{R})$ we have that

$$\lim_{\lambda \to 0} \frac{g(x+\lambda e_\ell,\omega) - g(x,\omega)}{\lambda} = \partial_{x^\ell} g(x,\omega), \tag{8.22}$$

for every $\omega \in \Omega$. Since the family (8.21) is uniformly integrable we have that the limit (8.22) is not only pointwise (in $\omega \in \Omega$) but also in $L^1(\Omega)$. Thus we get

$$\begin{aligned} \lim_{\lambda \to 0} \frac{G(x+\lambda e_\ell) - G(x)}{\lambda} &= \lim_{\lambda \to 0} \frac{\mathbb{E}[g(x+\lambda e_\ell,\omega) - g(x,\omega)]}{\lambda} \\ &= \mathbb{E}\left[\lim_{\lambda \to 0} \frac{g(x+\lambda e_\ell,\omega) - g(x,\omega)}{\lambda}\right] = \mathbb{E}[\partial_{x^\ell} g(x,\omega)], \end{aligned} \tag{8.23}$$

where we can exchange the limit with the expectation because the convergence (8.22) is in $L^1(\Omega)$. This proves the existence of the derivatives $\partial_{x^\ell} G(x)$. The fact that $\partial_{x^\ell} G(x)$ is continuous follows from the first part of the present lemma and the representation formula (8.23). □

Remark 8.16. If $g \in C^2(\mathbb{R}^k,\mathbb{R})$ and we have, for some $p>1$,

$$\sup_{x \in \mathcal{K}} \mathbb{E}[|\partial_{x^\ell x^{\ell'}} g(x,\omega)|^p] \leqslant C_{\mathcal{K}},$$

then $G \in C^2(\mathbb{R}^k,\mathbb{R})$. Indeed we can apply the Lemma 8.15 to the function $\partial_{x^\ell} G = \mathbb{E}[\partial_{x^\ell} g(x,\omega)]$.

Proof. The uniqueness of solution to equation (8.17) has been proved in Theorem 8.3.

Let $f \in C^2(\mathbb{R}^m,\mathbb{R})$ growing at most polynomially (say of degree $L \in \mathbb{N}$) as $x \to +\infty$, and consider the function

$$u(t,x) = \mathbb{E}[f(X_t^x)].$$

Consider $t,s \in \mathbb{R}_+$ and $t \geqslant s$, then

$$u(t,x) - u(s,x) = \mathbb{E}\left[\int_s^t \mathcal{L} f(X_\tau^x) \mathrm{d}\tau + \sum_{k=1}^m \sum_{j=1}^n \int_s^t \sigma_j^k(X_\tau^x) \partial_{x^k} f(X_\tau^x) \mathrm{d}B_\tau^j\right].$$

As usual, we can prove that $\int_s^t \sigma_j^k(X_\tau^x) \partial_{x^k} f(X_\tau^x) \mathrm{d}B_\tau^j$ are martingale, and so

$$u(t,x) - u(s,x) = \int_s^t \mathbb{E}[\mathcal{L} f(X_\tau^x)] \mathrm{d}\tau = \int_s^t G(\tau,x) \mathrm{d}\tau.$$

By Lemma 8.15 (and the polynomial growth of $\mathcal{L} f(X_\tau^x)$) is continuous the function $G(\tau,x)$ is continuous, and thus u is differentiable with respect to the time with continuous derivatives. Furthermore, by Corollary 7.15, and the polynomial growth of f, we have

$$|u(t,x)| \leqslant \mathbb{E}[|f(X_t^x)|] \leqslant K \mathbb{E}[(1+|X_t^x|^N)] \leqslant K\,(1 + C_N e^{C_N t}(1+|x|^N))$$

and also

$$|\partial_t u(t,x)| = |G(t,x)| \leqslant \mathbb{E}[|\mathcal{L} f(X_t^x)|] \leqslant K' \mathbb{E}[(1+|X_t^x|^{N+1})] \leqslant K\,(1 + C_{N+1} e^{C_{N+1} t}(1+|x|^{N+1})).$$

Furthermore, by Theorem 8.11, for any $t \in \mathbb{R}_+$ and $\omega \in \Omega$, the map $x \to f(X_t^x(\omega))$ is $C^2(\mathbb{R}^m,\mathbb{R})$, furthermore we have

$$\partial_{x^\ell}(f(X_t^x(\omega))) = \sum_{k=1}^m \partial_{y^k} f(X_t^x(\omega)) \xi_{\ell,t}^{x,k}$$

$$\partial_{x^\ell x^{\ell'}}(f(X_t^x(\omega))) = \sum_{k,k'=1}^m \partial_{y^k y^{k'}} f(X_t^x(\omega)) \xi_{\ell,t}^{x,k} \xi_{\ell,t}^{x,k'} + \sum_{k=1}^m \partial_{y^k} f(X_t^x(\omega)) \chi_{\ell,\ell',t}^{x,k}.$$

Thus we have that, for any $p>1$ and Corollary 8.13,

$$\begin{aligned} \mathbb{E}[|\partial_{x^\ell}(f(X_t^x(\omega)))|^p] &\leqslant m^{p-1} \sum_{k=1}^m \mathbb{E}[|\partial_{y^k} f(X_t^x(\omega))|^p |\xi_{\ell,t}^{x,k}|^p] \\ &\leqslant m^p 2^{p-1} K^p C^{\,p} e^{pCt} \mathbb{E}[(1+|X_t^x|^{Np})] \\ &\leqslant m 2^{p-1} K^p C^p \, e^{pCt}(1 + C_{pN} e^{C_{pN} t}(1+|x|^{pN})), \end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{E}[|\partial_{x^\ell x^{\ell'}}(f(X_t^x(\omega)))|^p] &\leqslant 2^{p-1}m^{2p-1}\sum_{k,k'=1}^{m}\mathbb{E}[|\partial_{y^k y^{k'}}f(X_t^x(\omega))\xi_{\ell,t}^{x,k}\xi_{\ell,t}^{x,k'}|^p]\\ &\quad +2^{p-1}m^{p-1}\sum_{k=1}^{m}\mathbb{E}[|\partial_{y^k}f(X_t^x(\omega))\chi_{\ell,\ell',t}^{x,k}|^p]\\ &\leqslant (2^p m^{2p+1}+2^p m^p)(K^p C^{2p}e^{2pCt}+K^p C^p e^{pCt})(1+C_{pN}e^{C_{pN}t}(1+|x|^{pN})).\end{aligned}$$

Thus we can apply Lemma 8.15 and Remark 8.16, obtaining that for any $t\in\mathbb{R}_+$, $u(t,x)\in C^2(\mathbb{R}^m,\mathbb{R})$ with at most polynomial growth in the derivatives. This implies that $u(t,x)\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^m,\mathbb{R})$. The thesis, thus, follows from Proposition 8.5. □

Bibliography

[1] Anton Bovier. Introduction to Stochastic Analysis. Lecture Uni Bonn WS 2020/21. https://www.dropbox.com/s/vyoa2qqyw4zsa8o/stochana.pdf?dl=0.

[2] Andreas Eberle. Introduction to Stochastic Analysis. Lecture Uni Bonn WS 2018/19. https://uni-bonn.sciebo.de/s/kzTUFff5FrWGAay.

[3] Hiroshi Kunita. *Stochastic flows and jump-diffusions*, volume 92 of *Probability Theory and Stochastic Modelling*. Springer, Singapore, 2019.

[4] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*, volume 274 of *Graduate Texts in Mathematics*. Springer, [Cham], French edition, 2016.

[5] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, Third edition, 1999.

[6] Walter Rudin. *Principles of mathematical analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, Third edition, 1976.

[7] Timo Seppäläinen. Basics of Stochastic Analysis. Lecture notes, Department of Mathematics, University of Wisconsin. Https://people.math.wisc.edu/ seppalai/courses/735/notes2014.pdf.

[8] Richard L. Wheeden and Antoni Zygmund. *Measure and integral*. Pure and Applied Mathematics, Vol. 43. Marcel Dekker, Inc., New York-Basel, 1977. An introduction to real analysis.

[9] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.

[10] Marc Yor. Sur quelques approximations d'intégrales stochastiques. In *Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976)*, pages 518–528. 1977.