Note 1 – Version 1 – 2022.5.3 – Please send remarks and corrections to gubinelli@iam.uni-bonn.de

Stochastic differential equations

We introduce various notions of solutions to a stochastic differential equation (SDE) driven by a Brownian motion: weak solutions, strong solutions and martingale solutions. Uniqueness in law and pathwise uniqueness are two relevant concepts associated to these solutions. The Yamada–Watanabe theorem links these notions one to the other. Regularity of coefficients allows to prove existence of strong solutions. Time change for diffusions and continuous martingales. One-dimensional diffusion can be completely characterised.

General organisation of the course

Lecures:

Thusday 16-18 (c.t.) and Thursday 12-14 (c.t.). We10/Kleiner Hörsaal. in presence.

Tutorials: Wed 8-10 SemR 0.007, Wed 12-14 SemR 0.007.

Song, Chunqui: s6chsong@uni-bonn.de
Liu, Min: s6miliuu@uni-bonn.de

Prerequisites

"Foundations/Introduction on Stochastic Analysis". Probability measures, continuous time stochastic processes, Kolmogorov's construction of stoch. proc., continuous time martingales, stochastic integration, Ito formula, SDE. Give a look at

https://www.iam.uni-bonn.de/abteilung-gubinelli/teaching/found-stoch-analysis-ws1920/

You have to be familiar to the following basic concepts: adapted process, continuous time martingale, local martingale, semimartingale, stochastic integral wrt. semimartingale, (one-)variation of a process, quadratic variation of a processs, co-variation, Riemann-Stiljest integral, Ito formula, Levy caracterisation of Brownian motion (in one dimension).

Introduction

Stochastic Analysis: set of tools to study stochastic (continuous) processes (i.e. Brownian motion, semimartingales, solutions to SDE, random fields).

Wiener '40 (Brown. mot., Lebesgue's theory)

Doob's/Levy/Ito ('40-'50) / Kunita/Watanabe/McKean/Malliavin/…

Malliavin derivative / White-noise calculus

Generalisation of analysis adapted to the study of stoch. proc.

Itô introduced SDE in the '40 with the aim of constructing diffusions, that is strong Markov processes with continuous paths and with generators which are second order differential operators. Stochastic analysis allows a pathwise approach to the construction of laws on path spaces and SDEs are the main tool for such constructions. SDE are also a natural approach to model physical systems which evolve in time and which are perturbed by "noise" (that is effect which we are not able to describe deterministically and for which we choose a probabilistic description). Recently, stochastic analysis has turned out to be a suitable tool to discuss mathematical finance (but Bachelier as early as the beginning of XX century introduced Brownian motion as a model of market prices). Stochastic analysis allows to easily remove the Markov hypothesis from the description of random processes (for example allowing memory in the coefficients) and more importantly allow to discuss the infinite dimensional situations more easily or in general certain Markov processes in very large state spaces (e.g. mean field models) in a relatively intuitive and direct fashion.

Main bibliographic references: [2, 3, 4].

Content of the course

- **Stoch. Diff. equations**: weak, strong, martingale problems. Links between the various notions. Including questions of uniqueness of solutions (pathwise uniq, weak uniq, uniq. of mart. problem).

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s.$$

- **Techniques for SDEs**. time-change ($X_t = Y_{f(t)}$), Girsanov's theorem ($\mathbb{Q} \ll \mathbb{P}$, $(\mathscr{F}_t)_t$), Tanaka's formula, conditioning (Doob's h-transform), singular conditioning (cond. on events of prob. zero). Doss–Sussmann technique (exact solutions to SDEs, link with control theory and ODE theory). Relation with PDE theory.

- **Martingale representation theorem.** (every mart. on a Brownian filtration is a stoch. integral). The formula of Boué–Dupuis ('90) - gives a variational formula for expectation values over a Brownian filtration. Large deviations for SDE:

$$\mathrm{d}X_t^\varepsilon = b(X_t^\varepsilon)\mathrm{d}t + \varepsilon\sigma(X_t^\varepsilon)\mathrm{d}B_t$$

$\varepsilon > 0$ small. $(X^\varepsilon)_{\varepsilon \geqslant 0}$ What happens for $\mu^\varepsilon(A) := \mathbb{P}(X^\varepsilon \in A)$ as $\varepsilon \to 0$. $\mu^\varepsilon \to \delta_{\text{ODE}}$. How fast is a question for large deviations theory.

$$\mu^\varepsilon(A) \approx \exp\left(-\frac{I(A)}{\varepsilon^2}\right), \qquad I(A) = \inf_{f \in A} I(f).$$

- **Diffusions on manifolds.** $(X_t \in \mathscr{M})_{t \geqslant 0}$ SDE??? Brownian motion on $\mathscr{M}$, relation with differential geometry. Δ Laplace–Beltrami.

- **Numerical methods for SDE.** $(X_t^n)_{t \geqslant 0}$ Euler-Maruyama method. Strong, weak approximations. As $n \to \infty$,

  $$\mathbb{E}(f(X_t)) \approx \mathbb{E}(f(X_t^n)), \qquad \mathbb{E}\|X_t - X_t^n\| \approx 0.$$

  Stochastic Taylor expansion (iterated stochastic integrals)

  $$f(B_t) = f(B_s) + f'(B_s)(B_t - B_s) + f''(B_s) \underbrace{\int_s^t \left(\int_s^u \mathrm{d}B_v\right) \mathrm{d}B_u}_{\mathbb{B}_{s,t}^2} + \cdots$$

- **Malliavin calculus** (?) (P. Malliavin '80) Analysis on infinite dimensional measure spaces. Wiener measure $\mathscr{W}(A) = \mathbb{P}(B \in A)$ $A \in \mathscr{B}(C([0,1]; \mathbb{R}))$. $\mathscr{W}$ is a probability measure on $C([0,1]; \mathbb{R})$. Wiener measure is a *replacement* for Lebesgue measure in $C([0,1]; \mathbb{R})$. Quasi-invariant under shift. Lebesgue/Sobolev type spaces on $C([0,1]; \mathbb{R})$. Notion of derivative: Malliavin derivative. Link to the martingale rep. theorem and to iterated stochastic integrals.

---

1 Notions of existence and uniqueness for SDEs

Setting. Probability space $(\Omega, \mathscr{F}, \mathbb{P})$, filtration $(\mathscr{F}_t)_{t \geqslant 0}$ right-continuous, $\mathbb{P}$-completed.

Definition 1. *A weak solution of the SDE in* $\mathbb{R}^n$

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t \in [0,T] \tag{1}$$

$$X_0 = x \in \mathbb{R}^n$$

is a pair of adapted processes (X, B) *where* $(B_t)_{t \geqslant 0}$ *is a m-dimensional Brownian motion and* b, σ *are coefficients* $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m; \mathbb{R}^n)$ *such that almost surely*

$$\int_0^t |b(X_s)| \mathrm{d}s < \infty, \quad \int_0^t \mathrm{Tr}(\sigma(X_s)\sigma(X_s)^T) \mathrm{d}s < \infty, \qquad t \in [0,T]$$

and that

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s, \qquad t \in [0,T].$$

Remark 2. Unless specified otherwise we will consider always continuous modifications of X and B. For B this always exists thanks to Kolmogorov's continuity criterion, for X this exists since X is a semimartingale whose martingale part is given by a stochastic integral wrt. the Brownian motion.

$\sigma = (\sigma_\alpha)_{\alpha=1,\ldots,m}$ family of vector-fields $\sigma_\alpha: \mathbb{R}^n \to \mathbb{R}^n$ (this is the right point of view on manifolds)

Control-theory point of view:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sum_{\alpha=1}^m \sigma_\alpha(X_t)\mathrm{d}B_t^\alpha.$$

$$\sum_{\alpha=1}^m \int_0^t |\sigma_\alpha(X_t)|^2 \mathrm{d}s < \infty.$$

Definition 3. *A* ***strong solution*** *to the SDE above is a weak solution such that X is adapted to the filtration* $(\mathscr{F}_t^{B,\mathbb{P}})_{t\geqslant 0}$ *generated by B and completed according to* $\mathbb{P}$, $\mathscr{F}_t^{B,\mathbb{P}} := \overline{\sigma(B_s: s\in[0,t])}^{\mathbb{P}}$.

$$X_t \hat{\in} \mathscr{F}_t \Rightarrow X_t(\omega) = \Phi_t((B_s(\omega))_{s\in[0,t]})$$

Facts.

- There are weak solutions which are not strong. (Tanaka's example, we will see it later on)
- There are SDEs which do not have strong solutions.
- A weak solution is really the data $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geqslant 0}, X, B)$.

Definition 4. *An SDE has* ***uniqueness in law*** *iff two solutions* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geqslant 0}, X, B)$ $(\Omega', \mathscr{F}', \mathbb{P}', (\mathscr{F}_t')_{t\geqslant 0}, X', B')$ *are such that*

$$\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{P}'}(X') \in \Pi(C([0,T];\mathbb{R}^n), \mathscr{B}(C([0,T];\mathbb{R}^n))).$$

Definition 5. *An SDE has* ***pathwise uniqueness*** *if for any two solutions* X, X' *defined on the same filt. prob. space and with the* ***same*** *BM B we have that they are indistinguishable, i.e.*

$$\mathbb{P}(\exists t \in [0,T]: X_t \neq X_t') = 0.$$

Some examples of all the possible situations

Example 6. [No existence] The following SDE on $\mathbb{R}$ has no weak solution

$$\mathrm{d}X_t = -\frac{1}{2X_t}\mathbb{1}_{X_t\neq 0}\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = 0. \tag{2}$$

Ito formula

$$X_t^2 = 2\int_0^t X_s \mathrm{d}X_s + \int_0^t \mathrm{d}s = -\int_0^t \mathbb{1}_{X_s\neq 0}\mathrm{d}s + 2\int_0^t X_s \mathrm{d}B_s + \int_0^t \mathrm{d}s = \int_0^t \mathbb{1}_{X_s=0}\mathrm{d}s + 2\int_0^t X_s \mathrm{d}B_s.$$

Since $[X]_t = t$ then the occupation time formula (which we assume for now, we will go back to this when discussing Tanaka's formula) we have

$$\int_0^t \mathbb{1}_{X_s=0}\mathrm{d}s = 0.$$

Therefore $(X_t^2)_t$ is a local martingale, which is positive and such that $X_0^2 = 0 \Rightarrow X_t = 0$ for all $t \in [0, T]$. But $X_t = 0$ is not a solution to the SDE (2).

Example 7. [No strong sol, nor pathwise uniqueness, ∃weak solutions, uniqueness in law] Tanaka's SDE:

$$\mathrm{d}X_t = \mathrm{sgn}(X_t)\mathrm{d}B_t, \quad X_0 = 0. \tag{3}$$

We will study this SDE later on in the course. It does not have strong solutions nor pathwise uniqueness, it has weak solutions and they all have the same law. In particular any weak solution is a Brownian motion.

Example 8. [No uniqueness, ∃strong]

$$\mathrm{d}X_t = \mathbb{1}_{X_t\neq 0}\mathrm{d}B_t, \quad X_0 = 0.$$

The process $X_t = 0$ is a solution but also the process $X_t = B_t$ is a solution, indeed in this second case we have $X_t - B_t = -\int_0^t \mathbb{1}_{X_s=0}\mathrm{d}B_s$ and this process has zero quadratic variation almost surely:

$$[X - B]_t = \int_0^t \mathbb{1}_{X_s=0}\mathrm{d}s = 0$$

by the *occupation time formula* since $\mathrm{d}[X]_t \ll \mathrm{d}t$. No pathwise-!, law-!. Assume on the probability space there is also a Bernoulli variable ξ (e.g. independent of B) assume $\mathscr{F}_0 \supseteq \sigma(\xi)$ and let

$$X_t(\omega) = \begin{cases} 0 & \text{if } \xi(\omega) = +1 \\ B_t(\omega) & \text{if } \xi(\omega) = 0 \end{cases}$$

This solution is not strong.

Example 9. [No strong sol. and no uniq.]

$$\mathrm{d}X_t = \mathbb{1}_{X_t \neq 1} \operatorname{sgn}(X_t) \mathrm{d}B_t, \quad X_0 = 0.$$

Here there exists weak solutions, no pathwise uniq., no strong solutions, no uniqueness in law. Indeed the Tanaka example Y is a solution but also $Z_t = Y_{t \wedge \tau}$ where $\tau = \inf\{t \geqslant 0: Y_t = 1\}$.

After these (counter-)examples we give now some general relations among the two concepts of existence and the two concepts of uniqueness for SDEs.

Theorem 10. (Yamada–Watanabe) *Weak existence+pathwise uniqueness $\Rightarrow$ strong existence*

(for the proof see also [5] or [3, Chap. IX,Thm. 1.7])

Theorem 11. *pathwise uniqueness $\Rightarrow$ uniqueness in law*

Uniqueness in law could be formulated with respect to the *joint law* of the pair (X, B). A result of Cherny [1] shows that the two concepts are equivalent and that together with existence of strong solutions they imply pathwise uniqueness.

Theorem 12. *(Cherny) Uniqueness in law implies uniqueness of the law of the pair (X, B), i.e.*

$$\mathrm{Law}_{\mathbb{P}}(X, B) = \mathrm{Law}_{\mathbb{P}'}(X', B').$$

Theorem 13. *(Cherny) Strong existence+uniqiess in law $\Rightarrow$ pathwise uniqueness*

So, overall, the situation is the following:

a) It may happen that there are no solution in any probability space;

b) if there exists a strong solution on a probability space then it is possible to construct solutions on any other probability space (carrying a Brownian motion). However there may be multiple solutions.

c) If pathwise uniqueness holds and there exists a solution on some probability space, then on any other probability space (carrying a Brownian motion) there exists only one solution and it is strong (Yamada–Watanabe).

d) The same ideal situation of point d) is reached if uniqueness in law holds and there exists a strong solution.

We are going to sketch the proofs of these facts.

Weak existence is usually obtained via approximations, apriori estimates and compactness arguments. Pathwise uniqueness is done by direct comparison of two solutions.

Proof. Of Theorem 11. Take two solutions $(\Omega, \mathbb{P}, \mathscr{F}, X, B)$, $(\Omega', \mathbb{P}', \mathscr{F}', X', B')$ we know pathwise uniqueness and we want to deduce $\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{P}'}(X')$. It would be easy if $(\Omega, \mathscr{F}) = (\Omega', \mathscr{F}')$ and $B = B'$ since then pathwise uniqueness applies and $X = X'$ from which follows that their laws are the same. Let almost surely

$$\rho_{B(\omega)}(A) = \mathbb{P}(X \in A|B)(\omega), \qquad \rho'_{B'(\omega')}(A) = \mathbb{P}'(X' \in A|B')(\omega'), \qquad A \in \mathscr{B}(\mathscr{C}^n)$$

with $\mathscr{C}^n = C(\mathbb{R}_+; \mathbb{R}^n)$. Both ρ, ρ' are regular conditional probabilities, i.e. probability kernels

$$\mathscr{C}^m \to \Pi(\mathscr{C}^n, \mathscr{B}(\mathscr{C}^n)).$$

This is possible since $\mathscr{C}^m$ is Polish. We can define a probabily measure $\mathbb{Q}$ on the filtered measure space $\tilde{\Omega} = \mathscr{C}^n \times \mathscr{C}^n \times \mathscr{C}^m$ with canonical process $(X_t, Y_t, B_t): \tilde{\Omega} \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ given by

$$\mathbb{Q}(\mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3) = \rho_{\omega_3}(\mathrm{d}\omega_1)\, \rho'_{\omega_3}(\mathrm{d}\omega_2)\, \mu(\mathrm{d}\omega_3)$$

where $\mu \in \Pi(\mathscr{C}^m)$ is the law of the Brownian motion in $\mathbb{R}^m$. Then easy to check that

$$\mathrm{Law}_{\mathbb{Q}}(X, B) = \mathrm{Law}_{\mathbb{P}}(X, B), \qquad \mathrm{Law}_{\mathbb{Q}}(Y, B) = \mathrm{Law}_{\mathbb{P}'}(X', B').$$

Technical point (that we will not prove here): the process $(\tilde{\Omega}, \mathbb{Q}, X, B)$ is a weak solution and $(\tilde{\Omega}, \mathbb{Q}, Y, B)$ is also a weak solution. Assuming this, by pathwise uniqueness we have that $X = Y$ almost surely which implies that

$$\mathrm{Law}_{\mathbb{P}}(X) = \mathrm{Law}_{\mathbb{Q}}(X) = \mathrm{Law}_{\mathbb{Q}}(Y) = \mathrm{Law}_{\mathbb{P}'}(X')$$

that is uniqueness in law. □

Proof of Theorem 10 (Yamada–Watanabe). We want to prove that there exists $\Phi: \mathscr{C}^m \to \mathscr{C}^n$ such that letting $Z = \Phi(B)$ we have that (Z, B) is a solution to the SDE. In this case we should have that its law is given by

$$\mathbb{P}((Z, B) \in (\mathrm{d}\omega_1 \times \mathrm{d}\omega_2)) = \delta_{\Phi(\omega_2)}(\mathrm{d}\omega_1)\, \mu(\mathrm{d}\omega_2), \qquad \omega_1 \in \mathscr{C}^n, \omega_2 \in \mathscr{C}^m$$

But the the previous argument give us that any two weak solutions have the same joint distributions, that is $\mathrm{Law}_{\mathbb{P}}(X, B) = \mathrm{Law}_{\mathbb{P}'}(X', B')$. Let call $\rho: \mathscr{C}^m \to \Pi(\mathscr{C}^n, \mathscr{B}(\mathscr{C}^n))$ the regulart conditional distribution under $\mathbb{P}$ of X given B. We need to prove that

$$\rho_{\omega_2} = \delta_{\Phi(\omega_2)}, \qquad \omega_2 \in \mathscr{C}^m,$$

that is, for *a.e.* ω_2 with respect to the Wiener measure μ we must have that ρ_{ω_2} is a Dirac mass at some point $\Phi(\omega_2)$. For this is enough to prove that $\rho_{\omega_2}(A) \in \{0,1\}$ for all $A \in \mathscr{B}(\mathscr{C}^m)$. Consider the measure $\mathbb{Q}$ above with $\rho' = \rho$, namely

$$\mathbb{Q}(\mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3) = \rho_{\omega_3}(\mathrm{d}\omega_1)\, \rho_{\omega_3}(\mathrm{d}\omega_2)\, \mu(\mathrm{d}\omega_3).$$

Then under $\mathbb{Q}$ both (X,B) and (Y,B) are weak solutions and by the assumption of pathwise uniqueness we deduce that $\mathbb{Q}(X = Y) = 1$ (also using continuity of the solutions). Therefore we have

$$0 = \mathbb{Q}(X \neq Y) = \int_{\tilde{\Omega}} \mathbb{1}_{\omega_1 \neq \omega_2} \rho_{\omega_3}(\mathrm{d}\omega_1)\, \rho_{\omega_3}(\mathrm{d}\omega_2)\, \mu(\mathrm{d}\omega_3).$$

By Fubini this implies

$$\int_{\mathscr{C}^n} \mathbb{1}_{\omega_1 \neq \omega_2} \rho_{\omega_3}(\mathrm{d}\omega_1) = 1,$$

for μ-a.e. $\omega_3 \in \mathscr{C}^m$

Now for μ-a.e. $\omega_3 \in \mathscr{C}^m$:

$$\rho_{\omega_3}(A) = \int_{\mathscr{C}^n} \int_{\mathscr{C}^n} \mathbb{1}_{\omega_1 \in A} \rho_{\omega_3}(\mathrm{d}\omega_1)\, \rho_{\omega_3}(\mathrm{d}\omega_2)$$

$$= \int_{\mathscr{C}^n} \int_{\mathscr{C}^n} \mathbb{1}_{\omega_1 \neq \omega_2} \mathbb{1}_{\omega_1 \in A} \rho_{\omega_3}(\mathrm{d}\omega_1)\, \rho_{\omega_3}(\mathrm{d}\omega_2)$$

$$= \int_{\mathscr{C}^n} \int_{\mathscr{C}^n} \mathbb{1}_{\omega_1 \neq \omega_2} \mathbb{1}_{\omega_1 \in A} \mathbb{1}_{\omega_2 \in A} \rho_{\omega_3}(\mathrm{d}\omega_1)\, \rho_{\omega_3}(\mathrm{d}\omega_2)$$

$$= \int_{\mathscr{C}^n} \int_{\mathscr{C}^n} \mathbb{1}_{\omega_1 \in A} \mathbb{1}_{\omega_2 \in A} \rho_{\omega_3}(\mathrm{d}\omega_1)\, \rho_{\omega_3}(\mathrm{d}\omega_2) = \rho_{\omega_3}(A)^2$$

therefore indeed $\rho_{\omega_3}(A) \in \{0,1\}$ for μ-a.e. $\omega_3 \in \mathscr{C}^m$. This implies that ρ_{ω_3} is a δ function □

Proof of Theorem 12. Theorem 12 is quite easy to prove if the SDE is one dimensional with $n = m = 1$ and $\sigma(x) > 0$ everywhere. Indeed observe that if (X,B) is a solution, then the process

$$M_t = \int_0^t \sigma(X_s) \mathrm{d}B_s = X_t - x - \int_0^t b(X_s) \mathrm{d}s \tag{4}$$

is a local martigale and it is measurable wrt. X. But then we have

$$\int_0^t (\sigma(X_s))^{-1} \mathrm{d}M_s = \int_0^t (\sigma(X_s))^{-1} \sigma(X_s) \mathrm{d}B_s = \int_0^t \mathrm{d}B_s = B_t$$

therefore B is X measurable and a consequence $B = \Psi(X)$ and we conclude that

$$\mathrm{Law}_{\mathbb{P}}(X,B) = \mathrm{Law}_{\mathbb{P}}(X, \Psi(X)) = \mathrm{Law}_{\mathbb{P}'}(X', \Psi(X')) = \mathrm{Law}_{\mathbb{P}'}(X', B')$$

if X, X' have the same law. Note that $B' = \Psi(X')$ because the map Ψ can be constructed in an almost sure way as follows. From (4) we have that there exists an (adapted) map Φ such that $M_t = \Phi_t(X)$. (and we will have the same for $M' = \Phi(X')$). And remember that for the stochastic integral $\int_0^t (\sigma(X_s))^{-1} \mathrm{d}M_s$ there exists a sequence of (deterministic) partitions $\Pi_n = \{t_1^n, \ldots, t_k^n, \ldots\}$ such that one can express $\int_0^t (\sigma(X_s))^{-1} \mathrm{d}M_s$ as almost sure limit of Riemann sums over the sequence of partitions

$$B_t = \int_0^t (\sigma(X_s))^{-1} \mathrm{d}M_s = \lim_n \sum_k (\sigma(X_{t_k^n}))^{-1} (M_{t_{k+1}^n} - M_{t_k^n}) = \lim_n \sum_k (\sigma(X_{t_k^n}))^{-1} (\Phi_{t_{k+1}^n}(X) - \Phi_{t_k^n}(X)) = \Psi_t(X)$$

and one can arrange to have the same partition for the primed solution and therefore have $B' = \Psi(X')$ at least $\mathbb{P}'$-a.s. (I skipped the detail of localizing the local martingale M in order to find the deterministic partition).

Let's discuss now the general case. Take $n \geqslant 1$, $m \geqslant 1$ $\sigma: \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m; \mathbb{R}^n) \approx \mathbb{R}^{n \times m}$.

Let $(\Omega^\sharp, \mathscr{F}^\sharp, \mathbb{P}^\sharp)$ another probability space on which there are two $\mathbb{R}^m$-Brownian motions $W, \bar{W}$. I form the product space $(\tilde{\Omega} = \Omega \times \Omega^\sharp, \tilde{\mathscr{F}} = \mathscr{F} \otimes \mathscr{F}^\sharp, \tilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}^\sharp)$ and on $\tilde{\Omega}$ I consider the solution (X, B) of the SDE together with processes $W, \bar{W}$. Note that $(W, \bar{W})$ is independent of (X, B). Of course $\mathrm{Law}_{\tilde{\mathbb{P}}}(X, B) = \mathrm{Law}_{\mathbb{P}}(X, B)$. For any fixed $x \in \mathbb{R}^n$ consider now $\varphi(x), \psi(x) \in \mathbb{R}^{m \times m}$ such that they are orthogonal projections on orthogonal subspaces:

$$\varphi(x) = \varphi(x)^T, \quad \psi(x) = \psi(x)^T, \quad \psi(x)^2 = \psi(x), \quad \varphi(x)^2 = \varphi(x), \quad \varphi(x)\psi(x) = 0, \quad \varphi(x) + \psi(x) = \mathbb{1}_{n \times n}$$

and such that $\sigma(x)\varphi(x) = \sigma(x)$ and $\sigma(x)\psi(x) = 0$. So $\mathrm{Im}(\varphi(x))^\perp = \mathrm{Ker}(\sigma(x)) = \mathrm{Im}(\psi(x))$. Now I define two new processes U, V on $\tilde{\Omega}$, with values in $\mathbb{R}^n$ and such that $U_0 = V_0 = 0$ and

$$\mathrm{d}U_t = \varphi(X_t)\mathrm{d}B_t + \psi(X_t)\mathrm{d}W_t$$
$$\mathrm{d}V_t = \psi(X_t)\mathrm{d}B_t + \varphi(X_t)\mathrm{d}\bar{W}_t$$

With this definition we have

$$\mathrm{d}[U^i, U^j]_t = \sum_{k,l} \varphi^{i,k}(X_t)\varphi^{j,l}(X_t)\underbrace{\mathrm{d}[B^k, B^l]_t}_{= \delta_{k,l} t} + \sum_{k,l} \varphi^{i,k}(X_t)\psi^{j,l}(X_t)\underbrace{\mathrm{d}[B^k, W^l]_t}_{=0}$$

$$+ \sum_{k,l} \psi^{i,k}(X_t)\varphi^{j,l}(X_t)\underbrace{\mathrm{d}[W^k, B^l]_t}_{=0} + \sum_{k,l} \psi^{i,k}(X_t)\psi^{j,l}(X_t)\underbrace{\mathrm{d}[W^k, W^l]_t}_{= \delta_{k,l}\mathrm{d}t}$$

$$= (\varphi(X_t)\varphi(X_t)^T)^{i,j}\mathrm{d}t + (\psi(X_t)\psi(X_t)^T)^{i,j}\mathrm{d}t = \delta_{i,j}\mathrm{d}t$$

by the properties of φ, ψ. Similarly $\mathrm{d}[V^i, V^j]_t = \delta_{i,j}\mathrm{d}t$ and moreover $\mathrm{d}[U^i, V^j]_t = 0$ since $\varphi(x)\psi(x) = 0$. We conclude the process (U, V) is a pair of independent $\mathbb{R}^m$-Brownian motions (by the multidimensional version of Levy's caracterisation theorem, we will prove it later on). Now we have

$$\int_0^t \sigma(X_s)\mathrm{d}B_s = \int_0^t \sigma(X_s)\varphi(X_s)\mathrm{d}B_s = \int_0^t \sigma(X_s)\mathrm{d}U_s.$$

This implies that $(\tilde{\Omega}, \tilde{\mathbb{P}}, (\tilde{\mathscr{F}}_t^{X,U})_{t\geqslant 0}, X, U)$ is a weak solution to the SDE.

I want to prove that V is independent of X. Define the filtration $(\mathscr{G}_t)_{t\geqslant 0}$ given by

$$\mathscr{G}_t = \sigma(U_s, X_s: s \leqslant t) \vee \sigma(V_s: s \geqslant 0).$$

Since U is independent of V, then U is still a $(\mathscr{G}_t)_{t\geqslant 0}$ Brownian motion, which implies in particular that $(U_t)_{t\geqslant 0}$ is independent of $\mathscr{G}_0$ therefore $(\tilde{\Omega}, \tilde{\mathbb{P}}, (\mathscr{G}_t)_{t\geqslant 0}, X, U)$ is still a solution of the SDE.

Now we want to consider the regular conditional probability of $\tilde{\mathbb{P}}$ given $\mathscr{G}_0$ that is the family of probability kernels $\mathbb{Q}: \tilde{\Omega} \to \Pi(\tilde{\Omega})$ such that

$$\mathbb{Q}_\omega(\cdot) = \tilde{\mathbb{P}}(\cdot|\mathscr{G}_0)(\omega), \qquad \text{for } \tilde{\mathbb{P}}\text{-a.e. } \omega \in \tilde{\Omega}.$$

I can do it because I can set up the full theorem in the case where $\tilde{\Omega}$ is the Polish space $\tilde{\Omega} = \mathscr{C}^{n+3m} = C(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$. The probability kernel $\mathbb{Q}$ is unique $\tilde{\mathbb{P}}$-a.s. Observe that $\mathscr{G}_0 = \sigma(V_s: s \geqslant 0)$ since we take a deteministic initial condition for $X_0 = x \in \mathbb{R}^n$.

Observe that almost sure events for $\tilde{\mathbb{P}}$ remains almost sure for $\mathbb{Q}_\omega$ (for $\tilde{\mathbb{P}}$-a.e. $\omega \in \tilde{\Omega}$), i.e.

$$1 = \tilde{\mathbb{P}}(A) \Rightarrow (\mathbb{Q}_\omega(A) = 1, \text{for } \tilde{\mathbb{P}}\text{-a.e. } \omega \in \tilde{\Omega})$$

indeed

$$1 = \tilde{\mathbb{P}}(A) = \int_{\tilde{\Omega}} \mathbb{Q}_\omega(A) \tilde{\mathbb{P}}(\mathrm{d}\omega).$$

By one of the theorems proven in Sheet 0, we have that $(\tilde{\Omega}, \mathbb{Q}_\omega, (\mathscr{G}_t)_{t\geqslant 0}, X, U)$ is still a weak solution to the SDE for $\tilde{\mathbb{P}}$-a.e. $\omega \in \tilde{\Omega}$. By uniqueness in law of the solutions to the SDE (by assumption), we have that the law under $\mathbb{Q}_\omega$ of X does not depend on ω, i.e.

$$\mathbb{Q}_\omega(X \in \cdot) = \mathrm{Law}_{\mathbb{Q}_\omega}(X) = \mathrm{Law}_{\mathbb{Q}_{\omega'}}(X) \qquad \text{for a.e. } \omega, \omega' \in \tilde{\Omega}.$$

Now

$$\tilde{\mathbb{P}}(X \in A, V \in B) = \int_{\{V\in B\}} \mathbb{Q}_\omega(X \in A) \tilde{\mathbb{P}}(\mathrm{d}\omega) = \int_{\tilde{\Omega}} \mathbb{Q}_{\omega'}(X \in A) \tilde{\mathbb{P}}(\mathrm{d}\omega') \int_{\{V\in B\}} \tilde{\mathbb{P}}(\mathrm{d}\omega) = \tilde{\mathbb{P}}(X \in A) \tilde{\mathbb{P}}(V \in B)$$

We conclude that X, V are independent.

Next we are going to prove that $B = B(X, V)$. Let us introduce a matrix $\chi(x) \in \mathbb{R}^{m\times n} \approx \mathscr{L}(\mathbb{R}^n; \mathbb{R}^m)$ such that $\chi(x)\sigma(x) = \varphi(x)$ (left inverse to σ) then let

$$M_t := X_t - x - \int_0^t b(X_s)\mathrm{d}s = \int_0^t \sigma(X_s)\mathrm{d}B_s$$

then M is local martingale and

$$\int_0^t \chi(X_s)\mathrm{d}M_s = \int_0^t \chi(X_s)\sigma(X_s)\mathrm{d}B_s = \int_0^t \varphi(X_s)\mathrm{d}B_s = \int_0^t \varphi(X_s)\mathrm{d}U_s$$

and

$$B_t = \int_0^t \underbrace{(\varphi(X_s) + \psi(X_s))}_{=1} \mathrm{d}B_s = \int_0^t \varphi(X_s)\mathrm{d}U_s + \int_0^t \psi(X_s)\mathrm{d}V_s = \int_0^t \chi(X_s)\mathrm{d}M_s + \int_0^t \psi(X_s)\mathrm{d}V_s$$

So we have that B_t can be expressed as a measurable function of (X, V). Therefore there exists a measurable map $\Gamma: \mathscr{C}^n \times \mathscr{C}^m \to \mathscr{C}^m$ such that $B = \Gamma(X, V)$. Therefore $(X, B) = (X, \Gamma(X, V))$. If (X', B') is another weak solution we will have in the same way that $(X', B') = (X', \Gamma(X', V'))$. But X is independent of V, V has the same law of V' (both m-dim BM) and X has the same law of X' (by assumption). So

$$\mathrm{Law}(X, V) = \mathrm{Law}(X', V')$$

and as a consequence

$$\mathrm{Law}(X, B) = \mathrm{Law}(X, \Gamma(X, V)) = \mathrm{Law}(X', \Gamma(X', V')) = \mathrm{Law}(X', B'). \qquad \square$$

Proof of Theorem 13. By strong existence there exist a weak solution (X, B) such that $X = \Phi(B)$. By uniqueness in law and the previous theorem we have that any two weak solutions (X, B) and (X', B') have the same law. So now take another weak solution (X', B) on the same probability space of (X, B) and with the same BM. Then we have that

$$\mathrm{Law}(X', B) = \mathrm{Law}(X, B) = \mathrm{Law}(\Phi(B), B)$$

It means that $\mathbb{P}(X' = \Phi(B)) = \mathbb{P}(X = \Phi(B)) = 1$ this implies that $\mathbb{P}(X = X') = 1$. So we have pathwise uniqueness. $\square$

---

2 Levy's caracterisation of multidimensional BM

Theorem 14. *Let $(M_t)_{t \geqslant 0}$ be a local martingale with values in $\mathbb{R}^n$ such that $M_0 = 0$ and*

$$[M^i, M^j]_t = \delta_{i,j} t \qquad t \geqslant 0,$$

then $(M_t)_{t \geqslant 0}$ is a $\mathbb{R}^n$-valued Brownian motion.

Proof. Take $v \in \mathbb{R}^n$ and let $M_t^v = \langle v, M_t \rangle$ a one dimensional local martingale. Note that

$$[M^v]_t = [M^v, M^v]_t = \sum_{i,j} v^i v^j [M^i, M^j]_t = \sum_{i,j} v^i v^j \delta_{i,j} t = \|v\|^2 t$$

Introduce the process

$$\Phi_t^\nu = \exp\left(iM_t^\nu + \frac{1}{2}[M_t^\nu]_t \right) = \exp\left(iM_t^\nu + \frac{1}{2}\|\nu\|^2 t \right) = \exp\left(\frac{1}{2}\|\nu\|^2 t \right)(\cos(M_t^\nu) + i\sin(M_t^\nu))$$

observe that

$$|\Phi_t^\nu| \leqslant \left| \exp\left(iM_t^\nu + \frac{1}{2}[M_t^\nu]_t \right) \right| \leqslant \exp\left(\frac{1}{2}\|\nu\|^2 t \right).$$

So the family $(\Phi_t^\nu)_{t\in[0,T]}$ is uniformly integrable in any bounded interval $[0,T]$. Moreover by Ito formula

$$\mathrm{d}\Phi_t^\nu = \exp\left(iM_t^\nu + \frac{1}{2}[M_t^\nu]_t \right)\left(i\mathrm{d}M_t^\nu + \frac{1}{2}\mathrm{d}[M_t^\nu]_t \right) + \underbrace{\frac{i^2}{2}\exp\left(iM_t^\nu + \frac{1}{2}[M_t^\nu]_t \right)\mathrm{d}[M^\nu]_t}_{\text{Ito correction}}$$

$$= \Phi_t^\nu \left(i\mathrm{d}M_t^\nu + \frac{1}{2}\cancel{\mathrm{d}[M_t^\nu]_t} - \frac{1}{2}\Phi_t^\nu \cancel{\mathrm{d}[M^\nu]_t} \right) = i\Phi_t^\nu \,\mathrm{d}M_t^\nu,$$

So we have

$$\Phi_t^\nu = \Phi_0^\nu + \int_0^t i\Phi_s^\nu \,\mathrm{d}M_s^\nu,$$

which shows that $(\Phi_t^\nu)_{t\in[0,T]}$ is a local martingale (because the stoch. int. $\int_0^t i\Phi_s^\nu \,\mathrm{d}M_s^\nu$ is a local mart.) and by the integrability is also a martingale (in that interval). Therefore

$$1 = (\Phi_s^\nu)^{-1}\Phi_s^\nu = (\Phi_s^\nu)^{-1}\mathbb{E}[\Phi_t^\nu|\mathscr{F}_s] = \mathbb{E}[\Phi_t^\nu(\Phi_s^\nu)^{-1}|\mathscr{F}_s]$$

$$= \mathbb{E}\left[\exp\left(i(M_t^\nu - M_s^\nu) + \frac{1}{2}([M_t^\nu]_t - [M_t^\nu]_s) \right)|\mathscr{F}_s \right]$$

which shows that

$$\mathbb{E}[\exp(i\langle \nu, M_t - M_s\rangle)|\mathscr{F}_s] = \exp\left(-\frac{\|\nu\|^2}{2}(t-s) \right)$$

for any $0 \leqslant s \leqslant t \leqslant T$ but since T is arbitrary, the relation is true for any time. First consequence of this relation is that $M_t - M_s$ is independent of $\mathscr{F}_s$ (because the conditional expectation of the complext exponential is non-random. Indeed for any $X \mathrel{\hat{\in}} \mathscr{F}_s$ (measurable wrt) one has

$$\mathbb{E}[\exp(i\langle \nu, M_t - M_s\rangle + i\alpha X)] = \mathbb{E}[\exp(i\langle \nu, M_t - M_s\rangle)]\mathbb{E}[i\alpha X]$$

(think about it) and by the properties of characteristic functions of vector valued r.v. one has that $M_t - M_s$ is independent of X. Moreover $M_t - M_s$ is a centred Gaussian vector with covariance matrix $\mathbb{1}_{n\times n}(t-s)$.

Using these two facts one prove by induction that for any $0 \leqslant t_1 < t_2 < \cdots < t_n$ we have that $(M_{t_{k+1}} - M_{t_k})_{k=1,\ldots,n-1}$ is an independent family of Gaussian vectors. Since $(M_t)_{t \geqslant 0}$ is continuous and adapted to $(\mathscr{F}_t)_{t \geqslant 0}$ we deduce that $(M_t)_{t \geqslant 0}$ is a n dimensional Brownian motion. $\square$

Some interesting facts come out of it.

Example 15. *(Random rotations)* Let B be a n-dimensional Brownian motion and $(O_t)_{t \geqslant 0}$ be an adapted process made of orthogonal transformations of $\mathbb{R}^n$, i.e. $O_t \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $O_t^T O_t = O_t O_t^T = \mathbb{1}_{n \times n}$. Then consider the $\mathbb{R}^n$ valued local martingale

$$M_t = \int_0^t O_s \mathrm{d}B_s = \left(\sum_{j=1}^n \int_0^t O_s^{i,j} \mathrm{d}B_s^j \right)_{i=1,\ldots,n}, \qquad \mathrm{d}M_t = O_t \mathrm{d}B_t$$

We have

$$[M^i, M^j]_t = \sum_{k,l=1}^n \int_0^t O_s^{i,k} O_s^{j,l} \underbrace{\mathrm{d}[B^k, B^l]_s}_{\delta_{k,l}\mathrm{d}s} = \int_0^t \sum_{k=1}^n O_s^{i,k} O_s^{j,k} \mathrm{d}s = \delta_{i,j} t$$

so by Levy's theorem this process is again a Brownian motion.

Example 16. *(Bessel process)* Let B be n-dimensional Brownian motion starting from $B_0 = x \in \mathbb{R}^n \neq 0$ and consider the process $R_t = |B_t|$ be the Euclidean length of B_t. I want to compute the dynamics of R_t. The function $\varphi(x) = |x|$ is smooth away from the origin and

$$\nabla \varphi(x) = \frac{x}{|x|}, \qquad \nabla^i \nabla^j \varphi(x) = \frac{\delta_{i,j}}{|x|} - \frac{x^i x^j}{|x|^3}, \qquad \mathbb{R}^d \ni x \neq 0.$$

By Ito formula

$$\mathrm{d}R_t = \mathrm{d}\varphi(B_t) = \sum_{i=1}^n \nabla^i \varphi(B_t) \mathrm{d}B_t^i + \frac{1}{2} \sum_{i,j=1}^n \nabla^i \nabla^j \varphi(B_t) \mathrm{d}[B^i, B^j]_t = \underbrace{\sum_{i=1}^n \frac{B_t^i}{|B_t|} \mathrm{d}B_t^i}_{=:\mathrm{d}W_t} + \frac{n-1}{2} \frac{1}{|B_t|} \mathrm{d}t = \mathrm{d}W_t + \frac{n-1}{2} \frac{\mathrm{d}t}{R_t}$$

as least for some small random time interval (in order to be sure that B_t does not touch the origin). Moreover the local martingale $(W_t)_t$ is really a Brownian motion, indeed

$$[W]_t = \int_0^t \sum_{i,j=1}^n \frac{B_s^i}{|B_s|} \frac{B_s^j}{|B_s|} \underbrace{\mathrm{d}[B^i, B^j]_s}_{=\delta_{i,j}\mathrm{d}s} = \int_0^t \mathrm{d}t = t.$$

So (R_t, W_t) is a weak solution of the one dimensional SDE

$$\mathrm{d}R_t = \frac{n-1}{2} \frac{\mathrm{d}t}{R_t} + \mathrm{d}W_t$$

with initial condition $R_0 = |B_0| > 0$. Observe that $R_t > 0$ for any time $t < T_0 = \inf\{t > 0: R_t = 0\}$. Here n has to be integer. But the SDE has a meaning also for $n \in \mathbb{R}$. From the properties of the Brownian motion we know that if $n \geqslant 2$ then $T_0 = +\infty$ almost surely, while if $n = 1$ then $T_0 < \infty$ a.s. What about uniqueness of solutions.

Theorem 17. *For pathwise uniqueness in one dimension see the theorem of Yamada-Watanabe in the Sheet 0, essentially we have pathwise uniqueness as soon as the drift $b: \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous i.e.*

$$|b(x) - b(y)| \leqslant C|x - y|$$

(same as for ODEs) and the diffusion coefficient $\sigma: \mathbb{R} \to \mathbb{R}$ is locally $1/2$-Hölder continuous, i.e.

$$|\sigma(x) - \sigma(y)| \leqslant C|x - y|^{1/2}.$$

Theorem 18. *In general dimension pathwise uniqueness holds when both b, σ are locally Lipschitz continuous (sufficient only).*

Therefore the SDE

$$\mathrm{d}R_t = \frac{n-1}{2}\frac{\mathrm{d}t}{R_t} + \mathrm{d}W_t,$$

has pathwise uniqueness away from 0, meaning that given two continuous solutions R, R' with same W and $R_0 = R_0' > 0$ and letting

$$T = \inf\{t \geqslant 0: R_t = 0 \text{ or } R_t' = 0\}$$

then $R_t = R_t'$ for all $t < T$. Indeed in any open set away from 0 the coefficients $\sigma(x) = 1$ and $b(x) = (n-1)/(2x)$ are locally Lipshitz. Which means that the unique strong solution stay positive when $n \geqslant 2$ and that $T_0 = +\infty$ a.s. The process $(R_t)_{t \geqslant 0}$ is called the n-dimensional Bessel process.

3 Martingale solutions

Martingale solutions is another technique to caracterise and study solutions of SDEs:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \underbrace{\sigma(X_t)\mathrm{d}B_t}_{\mathrm{d}M_t} \tag{5}$$

In this relation we need to discuss two processes: the solution X and the driving Brownian motion B. We take X to be $\mathbb{R}^n$-valued and B to be $\mathbb{R}^m$-valued and $b: \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m; \mathbb{R}^n)$ measurable and locally bounded.

Martingale solution characterise the process X alone without the need of introducing the driving BM. To start observe the following two facts: if X solve the SDE then

$$M_t = X_t - X_0 - \int_0^t b(X_s)\mathrm{d}s$$

is a local martingale with quadratic variation

$$[M^i, M^j]_t = \int_0^t a^{ij}(X_s)\mathrm{d}s, \qquad i,j = 1,\ldots,n$$

with $a(x) = \sigma(x)\,\sigma(x)^T$ i.e. $a(x)^{i,j} = \sum_{k=1}^m \sigma(x)^{i,k}\sigma(x)^{j,k}$. Similarly for any $f \in C^2(\mathbb{R}^n)$ by Ito formula we have

$$f(X_t) = f(X_0) + \underbrace{\int_0^t \nabla f(X_s)\cdot\sigma(X_s)\mathrm{d}B_s}_{=:M_t^f \text{ (local martingale)}} + \int_0^t \mathscr{L}f(X_s)\mathrm{d}s,$$

with $\mathscr{L}$ a linear operator (*generator*) defined on C^2 functions as

$$\mathscr{L}f(x) = b(x)\cdot\nabla f(x) + \frac{1}{2}\sum_{i,j=1}^n a^{i,j}(x)\nabla_i\nabla_j f(x).$$

Definition 19. *We say that $(X_t)_t$ is martingale solution of the SDE (5) if one of the following equivalent facts holds:*

a) For any $f \in C^2(\mathbb{R}^n)$ we have that

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)\mathrm{d}s$$

is a local martingale.

b) The $\mathbb{R}^n$-valued continuous process

$$M_t = X_t - X_0 - \int_0^t b(X_s)\mathrm{d}s$$

is a local martingale with covariation

$$[M^i, M^j]_t = \int_0^t a^{ij}(X_s)\mathrm{d}s,$$

c) For any $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ we have

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left[\left(\tfrac{\partial}{\partial s} + \mathscr{L}\right)f\right](s, X_s)\mathrm{d}s$$

is a local martingale.

This formulation is really a description of the law of X. Consider $\mathscr{C} = C(\mathbb{R}_+; \mathbb{R}^n)$ with the Borel σ-field $\mathscr{G}$, the canonical filtration $(\mathscr{G}_t)_{t\geqslant 0}$ and canonical process $Z_t(\omega) = \omega_t$.

Definition 20. *A probability $\mathbb{P}$ on $(\mathscr{C}, \mathscr{G}, (\mathscr{G}_t)_{t\geqslant 0})$ is a martingale solution (or a solution of the martingale problem) of the SDE if the canonical process $(Z_t)_t$ is a martingale solution under $\mathbb{P}$.*

Note that the notion of martingale depends on $\mathbb{P}$.

Theorem 21. *If $(\mathscr{C},\mathscr{G},(\mathscr{G}_t)_{t\geqslant 0},\mathbb{Q})$ is a martingale solution to the SDE (5) iff there exist a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and two processes (X,B) over it such that (X,B) is a weak solution to the SDE and* $\mathrm{Law}_{\mathbb{P}}(X)=\mathrm{Law}_{\mathbb{Q}}(Z)$.

Proof. If (X,B) is a weak solution then taking $\mathbb{Q}=\mathrm{Law}_{\mathbb{P}}(X)$ give that $\mathbb{Q}$ is a martingale solution. The more difficult part is to start from a solution of the martingale problem $\mathbb{Q}$ and try to reconstruct a Brownian motion B and then a weak solution. (this reminds us the situation in Cherny's theorem). Indeed if σ is non-degenerate, i.e. there exists a (locally bounded) two side inverse $\sigma(x)^{-1}$ then we could simply take the $\mathbb{Q}$-local martingale

$$M_t=Z_t-Z_0-\int_0^t b(Z_s)\mathrm{d}s$$

(recall that we are on $\mathscr{C}$ with canonical process Z) and define on $\mathscr{C}$

$$B_t:=\int_0^t \sigma(Z_s)^{-1}\mathrm{d}M_s$$

and check that this is indeed a $(\mathbb{Q},(\mathscr{G}_t)_{t\geqslant 0})$-Brownian motion and that

$$Z_t-Z_0-\int_0^t b(Z_s)\mathrm{d}s=\int_0^t \mathrm{d}M_s=\int_0^t \sigma(Z_s)\mathrm{d}B_s$$

so (Z,B) is a weak-solution. In this case we can perform the construction on the same probability space. If σ is not invertible we proceed as in Cherny's theorem. We have a left inverse $\chi(x)$ such that $\chi(x)\sigma(x)=\varphi(x)$ where $\varphi(x)$ is the orthogonal projection on $\ker(\varphi(x))^{\perp}$ and we call $\psi(x)$ the orthogonal projection on $\ker(\sigma(x))$. Now we take a larger probability space $(\Omega,\mathscr{F},\mathbb{P})$ with processes (X,W) such that $\mathrm{Law}_{\mathbb{P}}(X)=\mathrm{Law}_{\mathbb{Q}}(Z)$ and X,W are independent and W is a m-dimensional Brownian motion. The we set

$$M_t=X_t-X_0-\int_0^t b(X_s)\mathrm{d}s$$

$$B_t=\int_0^t \chi(X_s)\mathrm{d}M_s+\int_0^t \psi(X_s)\mathrm{d}W_s$$

now is easy to check that $(M_t)_t$ is a local martingale with quadratic variation $\mathrm{d}[M^i,M^j]_t=a^{i,j}(X_t)\mathrm{d}t$ (because under $\mathbb{P}$ the process X is a martingale solution) and moreover $(B_t)_t$ is a Brownian motion.

$$\mathrm{d}[B,B]_t=\chi(X_s)a(X_s)\chi(X_s)^T\mathrm{d}s+\psi(X_s)\psi(X_s)^T\mathrm{d}s$$

$$=\underbrace{\chi(X_s)\sigma(X_s)}_{\varphi(X_s)}\sigma(X_s)^T\chi(X_s)^T\mathrm{d}s+\psi(X_s)\psi(X_s)^T\mathrm{d}s$$

$$=(\varphi(X_s)\varphi(X_s)^T+\psi(X_s)\psi(X_s)^T)\mathrm{d}s=\mathbb{1}_{n\times n}\mathrm{d}s.$$

It remains to verify that

$$\tilde{M}_t:=\int_0^t \sigma(X_s)B_s=\int_0^t \sigma(X_s)\chi(X_s)\mathrm{d}M_s+\int_0^t \underbrace{\sigma(X_s)\psi(X_s)}_{=0}\mathrm{d}W_s=\int_0^t \sigma(X_s)\chi(X_s)\mathrm{d}M_s$$

coincides with M_t. This is a bit tricky since we do not have that $\sigma(X_s)\chi(X_s) = \mathbb{I}_{n\times n}$ given the fact that the image of $\sigma(x)$ could be smaller than the whole space. We have to use the specific form of the quadratic variation of M. It is clear that to show that $M = \tilde{M}$ is enough to show that $[M^i - \tilde{M}^i]_t = 0$ for all $t \geqslant 0$ and $i = 1, \ldots, n$ (why?). Moreover the quadratic variation is a quadratic functional, so it will be enough to show that

$$[M^i, \tilde{M}^j]_t = [\tilde{M}^i, \tilde{M}^j]_t = [M^i, M^j]_t = \int_0^t a^{ij}(X_s)\mathrm{d}s$$

which is now the result of a direct computation, for example:

$$[M^i, \tilde{M}^j]_t = \sum_\ell \int_0^t (\sigma(X_s)\chi(X_s))^{j,\ell}\mathrm{d}[M^i, M^\ell]_s = \sum_\ell \int_0^t (\sigma(X_s)\chi(X_s))^{j,\ell}(\sigma(X_s)\sigma(X_s)^T)^{i\ell}\mathrm{d}s$$

$$= \int_0^t (\sigma(X_s)\chi(X_s)\sigma(X_s)\sigma(X_s)^T)^{j,i} = \int_0^t (\sigma(X_s)\varphi(X_s)\sigma(X_s)^T)^{j,i} = \int_0^t (\sigma(X_s)\sigma(X_s)^T)^{j,i} = \int_0^t a^{ij}(X_s)\mathrm{d}s$$

and a similar computation for $[\tilde{M}^i, \tilde{M}^j]_t$ (check!). This conclude the proof that (X, B) is indeed the weak solution we were looking for. □

Remark 22. Note that the notion of martingale solution makes sense also when $\sigma = 0$. Exercise: prove that in this case a process X satisfies the martingale problem iff X is a solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t = b(X_t), \qquad t \geqslant 0.$$

In this case $\mathscr{L}f(x) = b(x)\cdot\nabla f(x)$.

See the book of Rogers and Williams and of Ethier and Kurtz for nice applications of the martingale problem approach.

Mart. prob. were introduced by Stroock and Varadhan (see their book: “Multidimensional diffusion processes”)

Remark 23. Uniqueness in law is equivalent to the uniqueness of solutions to the martingale problem.

Remark 24. The notion of martingale problem makes sense even when the process X does not take values in a vector space, e.g. on a manifold $\mathscr{M}$. Indeed note that X solve the martingale problem iff

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)\mathrm{d}s$$

is a local (real-valued) martingale for any $f \in C^2(\mathscr{M})$ where

$$\mathscr{L}f = Bf + \frac{1}{2}\sum_{\alpha=1}^m V_\alpha(V_\alpha f)$$

with $B, (V_\alpha)_{\alpha=1,\ldots,n}$ are vector fields on $\mathscr{M}$. In the case where $\mathscr{M} = \mathbb{R}^n$ we have

$$Bf = b(x)\cdot\nabla f(x) - \frac{1}{2}\sum_{\alpha=1}^n (\sigma_\alpha(x)\cdot\nabla\sigma_\alpha(x))\cdot\nabla f(x), \qquad V_\alpha f = \sigma_\alpha(x)\cdot\nabla f(x)$$

with $(\sigma_\alpha(x))_{\alpha=1,\ldots,m}$ the rows of the matrix $\sigma(x)$. We will discuss more on detail this application when we are going to study SDE on manifolds and Stratonovich integral.

4 Time change in martingale problems

Let X be the solution of the SDE (5) in the sense of martingale problem. Let $\varrho(x)\colon \mathbb{R}^n \to \mathbb{R}_+$ which is locally bounded and $\varrho(x) > 0$ everywhere and let

$$A_t := \int_0^t \varrho(X_s)\mathrm{d}s.$$

This process is increasing and continuous assume for simplicity that $A_\infty = +\infty$. Define $T_a = \inf\{t \geqslant 0\colon A_t \geqslant a\}$ for $a \geqslant 0$. Then $T_0 = 0$ and T is the left inverse to A in the sense that

$$T_{A_t} = \inf\{s \geqslant 0\colon A_s \geqslant A_t\} = t$$

for all $t > 0$ and is defined for all $a > 0$. Moreover T_a is a stopping time for the filtration generated by $(X_s)_s$. Define the process $Y_a = X_{T_a}$ for all $a > 0$. Now the question is to characterise $(Y_a)_{a\geqslant 0}$. Of course $Y_0 = X_0$. Take $f \in C^2(\mathbb{R}^n)$ and note that

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)\mathrm{d}s$$

is a local $(\mathscr{F}_t)_{t\geqslant 0}$-martingale. Therefore the process $(N_a^f)_{a\geqslant 0}$ defined by $N_a^f = M_{T_a}^f$ is a local martingale wrt. the filtraltrion $(\mathscr{G}_a)_{a\geqslant 0}$ defined by $\mathscr{G}_a = \mathscr{F}_{T_a}$ (recall the def of σ-algebra of a stopping time and the optional sampling theorem for continuous martingales with bounded quadratic variation, note also that $T_a \leqslant T_b$ if $a \leqslant b$).

$$N_a^f = M_{T_a}^f = f(X_{T_a}) - f(X_0) - \int_0^{T_a} \mathscr{L}f(X_s)\mathrm{d}s = f(Y_a) - f(Y_0) - \int_0^{T_a} \mathscr{L}f(X_s)\mathrm{d}s.$$

To conclude observe that by doing the change of variables $s = s(b)$ such that $b = A_s$ or $T_b = s$

$$\mathrm{d}b = \varrho(X_s)\mathrm{d}s$$

therefore

$$\mathrm{d}s = \frac{\mathrm{d}b}{\varrho(X_s)}$$

and

$$\int_0^{T_a} \mathscr{L}f(X_s)\mathrm{d}s = \int_0^a \mathscr{L}f(Y_b)\mathrm{d}s(b) = \int_0^a \mathscr{L}f(Y_b)\frac{\mathrm{d}b}{\varrho(X_s)} = \int_0^a \mathscr{L}^\varrho f(Y_b)\mathrm{d}b$$

with

$$\mathscr{L}^\varrho f(x) = \frac{1}{\varrho(x)}\mathscr{L}f(x) = \underbrace{\frac{1}{\varrho(x)}b(x)}_{b^\varrho} \cdot \nabla f(x) + \frac{1}{2}\sum_{i,j=1}^n \underbrace{\frac{1}{\varrho(x)}a^{i,j}(x)}_{=\sigma^\varrho(\sigma^\varrho)^T} \nabla_i \nabla_j f(x).$$

So $(Y_a)_{a\geqslant 0}$ solves the martingale problem associated to the generator $\mathscr{L}^\varrho$ namely, is associated to the SDE

$$\mathrm{d}Z_a = b^\varrho(Z_a)\mathrm{d}a + \sigma^\varrho(Z_a)\mathrm{d}B_a, \qquad a \geqslant 0$$

where $b^{\varrho}(x) = b(x) / \varrho(x)$ and $\sigma^{\varrho}(x) = \sigma(x) / (\varrho(x))^{1/2}$.

5 Uniqueness of the martingale problem for a diffusion

$\mathscr{C} = \mathscr{C}^n = C(R_+, \mathbb{R}^n)$ with its Borel σ-algebra $\mathscr{F}$ and canonical process $(X_t)_{t\geqslant 0}$ with associated filtration $(\mathscr{F}_t)_{t\geqslant 0}$. Remember that with $\Pi(\mathscr{C})$ we denote the probability measures on the path space $\mathscr{C}$.

Consider the generator $\mathscr{L}$ defined for any $f \in C^2(\mathbb{R}^n)$ as

$$\mathscr{L}f(x) = b(x) \cdot \nabla f(x) + \frac{1}{2}\mathrm{Tr}(a\nabla^2 f(x)), \qquad x \in \mathbb{R}^n,$$

with measurable and bounded coefficients.

Definition 25. *We say that $\mathbb{P}$ on $(\mathscr{C}, (\mathscr{F}_t)_{t\geqslant 0})$ is a solution of the martingale problem for the generator $\mathscr{L}$ iff for any $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$*

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + \mathscr{L}) f(s, X_s) \mathrm{d}s \tag{6}$$

is a $\mathbb{P}$-martingale wrt. $(\mathscr{F}_t)_{t\geqslant 0}$.

We want to discuss the uniqueness of such solutions, meaning the following.

Definition 26. *We say that the martingale problem (6) has unique solution if any two solutions $\mathbb{P}$, $\mathbb{Q} \in \Pi(\mathscr{C})$ of the martingale problem such that $\mathrm{Law}_{\mathbb{P}}(X_0) = \mathrm{Law}_{\mathbb{Q}}(X_0)$ then $\mathbb{P} = \mathbb{Q}$.*

This notion corresponds directly with the uniqueness in law of the corresponding weak solutions. It is enough that $\mathbb{P}, \mathbb{Q}$ coincide on finite dimensional distributions.

Let use observe that if $\varphi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ is a solution to the (parabolic) PDE (Kolmogorov backward equation)

$$\partial_t \varphi(t, x) = \mathscr{L}\varphi(t, x), \qquad t \geqslant 0, x \in \mathbb{R}^n, \tag{7}$$

Note that $(\partial_s + \mathscr{L})\varphi(r - s, X_s) = 0$ for any $r > s$, therefore for any $r > 0$ and any $t \in [0, r]$ the process

$$M_t^r = \varphi(r - t, X_t) - \varphi(r, X_0) - \int_0^t (\partial_s + \mathscr{L})\varphi(r - s, X_s)\mathrm{d}s = \varphi(r - t, X_t) - \varphi(r, X_0)$$

is a martingale under any solution $\mathbb{P}$ of the martingale problem associated to $\mathbb{P}$. Now $M_r^r = \varphi(0, X_r) - \varphi(r, X_0)$ so

$$0 = \mathbb{E}_{\mathbb{P}}[M_r^r - M_t^r | \mathscr{F}_t] = \mathbb{E}_{\mathbb{P}}[\varphi(0, X_r) - \varphi(r - t, X_t) | \mathscr{F}_t]$$

tells me that for any $r \geqslant t$ we have

$$\mathbb{E}_{\mathbb{P}}[\varphi(0, X_r) | \mathscr{F}_t] = \mathbb{E}_{\mathbb{P}}[\varphi(r - t, X_t) | \mathscr{F}_t] = \varphi(r - t, X_t), \qquad \mathbb{P} - a.s.$$

So the value of this expectation essentially do not depends on which solution of the martingale problem we get

$$\mathbb{E}_{\mathbb{P}}[\varphi(0,X_r)] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\varphi(0,X_r)|\mathscr{F}_0]] = \mathbb{E}_{\mathbb{P}}[\varphi(r,X_0)]$$

and if $\mathbb{Q}$ is another solution with $\mathrm{Law}_{\mathbb{Q}}(X_0) = \mathrm{Law}_{\mathbb{P}}(X_0)$ then we conclude that

$$\mathbb{E}_{\mathbb{P}}[\varphi(0,X_r)] = \mathbb{E}_{\mathbb{Q}}[\varphi(0,X_r)]$$

for any $r \geqslant 0$. Let us assume know that the Kolmogorov backward equation has solution for any initial condition $\psi \in C_0^\infty(\mathbb{R}^n)$ (where the 0 means compactly supported). This implies that if we use such solutions in the argument above we get that for any $\psi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\mathbb{E}_{\mathbb{P}}[\psi(X_r)] = \mathbb{E}_{\mathbb{Q}}[\psi(X_r)]$$

and this implies that

$$\mathrm{Law}_{\mathbb{P}}(X_r) = \mathrm{Law}_{\mathbb{Q}}(X_r) \tag{8}$$

for any $r \geqslant 0$. So we deduced that the one time marginals of $\mathbb{P}$ and $\mathbb{Q}$ coincide. Now let $\psi \in C_0^\infty(\mathbb{R}^n)$ and let φ^ψ to be the solution of (7) such that $\varphi(0,x) = \psi(x)$ for all $x \in \mathbb{R}^n$ then as we already seen $\mathbb{E}_{\mathbb{P}}[\psi(X_r)|\mathscr{F}_t] = \varphi^\psi(r-t,X_t)$, therefore for any $r_1 > r_2 \geqslant 0$ we have for any bounded and measurable $g: \mathbb{R}^n \to \mathbb{R}^n$

$$\mathbb{E}_{\mathbb{P}}[\psi(X_{r_1})g(X_{r_2})] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\psi(X_{r_1})|\mathscr{F}_{r_2}]g(X_{r_2})] = \mathbb{E}_{\mathbb{P}}[\underbrace{\varphi^\psi(r_1-r_2,X_{r_2})g(X_{r_2})}_{\tilde{g}(X_{r_2})}]$$

$$\underset{\text{eq.(8)}}{=} \mathbb{E}_{\mathbb{Q}}[\varphi^\psi(r_1-r_2,X_{r_2})g(X_{r_2})] = \mathbb{E}_{\mathbb{Q}}[\psi(X_{r_1})g(X_{r_2})]$$

since ψ and g are arbitrary we conclude that

$$\mathrm{Law}_{\mathbb{P}}(X_{r_1},X_{r_2}) = \mathrm{Law}_{\mathbb{Q}}(X_{r_1},X_{r_2}).$$

We can continue by induction and deduce that $\mathbb{P},\mathbb{Q}$ have the same finite dimensional marginals, and therefore are equal as probability measures on $\mathscr{C}$. (think about it). Moreover note that we also have for any $r > t$

$$\mathbb{E}_{\mathbb{P}}[\psi(X_r)|\mathscr{F}_t] = \varphi^\psi(r-t,X_t),$$

which implies that the process $(X_t)_{t\geqslant 0}$ under $\mathbb{P}$ is a Markov process, indeed for any $t_1 < \cdots < t_n < r$ we have

$$\mathbb{E}_{\mathbb{P}}[\psi(X_r)g(X_{t_1},\ldots,X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}[\psi(X_r)|\mathscr{F}_{t_n}]g(X_{t_1},\ldots,X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\varphi^\psi(r-t_n,X_{t_n})g(X_{t_1},\ldots,X_{t_n})]$$

but also

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}[\psi(X_r)|X_{t_n}]g(X_{t_1},\ldots,X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\varphi^\psi(r-t_n,X_{t_n})g(X_{t_1},\ldots,X_{t_n})]$$

from which we get

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}[\psi(X_r)|X_{t_n}]g(X_{t_1},\ldots,X_{t_n})] = \mathbb{E}_{\mathbb{P}}[\psi(X_r)g(X_{t_1},\ldots,X_{t_n})]$$

and by a monotone class argument one deduce that

$$\mathbb{E}[\psi(X_r)|X_{t_n}] = \mathbb{E}[\mathbb{E}[\psi(X_r)|X_{t_n}]|\mathscr{F}_{t_n}] = \mathbb{E}[\psi(X_r)|\mathscr{F}_{t_n}]$$

for any $\psi \in C_0^\infty(\mathbb{R}^n)$ which approximates any continuous function and then also indicator functions of open sets from which we conclude that it is true for any ψ which is bounded and measurable. This proves the Markov property of $(X_t)_{t\geqslant 0}$ under $\mathbb{P}$.

Theorem 27. *Assume that the Kolmogorov backward PDE*

$$\partial_t \varphi(t,x) = \mathscr{L}\varphi(t,x), \qquad \varphi(0,\cdot) = \psi$$

has a solution $\varphi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ for any $\psi \in C_0^\infty(\mathbb{R}^n)$ then the martingale problem associated to $\mathscr{L}$ in the sense of Definition 25 has a unique solution in the sense of Definition 26. (and as a consequence uniqueness of weak solutions to the associated SDE).

Remark 28. This reduces the uniqueness problem to a problem about *existence* of enough regular solutions to a PDE. Note that the set of initial conditions $C_0^\infty(\mathbb{R}^n)$ could be replaced by any set $\mathscr{D}$ with the property that if two probability measures $\mu, \nu \in \Pi(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} f(x)\,\mu(\mathrm{d}x) = \int_{\mathbb{R}^n} f(x)\,\nu(\mathrm{d}x), \qquad f \in \mathscr{D}$$

then $\mu = n$, i.e. $\mathscr{D}$ is a determining (or separating) class for $\Pi(\mathbb{R}^n)$.

Remark 29. What about existence of solutions to the martingale problem.

a) (Construction of the weak solution SDE) maybe strong solutions via fixpoint arguments, or time-change, or Girsanov transformation (to be seen), Doob's transform.

b) (Compactness arguments) Assume that we have a sequence of probabilities $(\mathbb{P}^n)_n$ on $\mathscr{C}$ such that $\mathbb{P}^n$ solve the martingale problem wrt. $\mathscr{L}^n$ (some generator). Assume also that we can show pointwise convergence of $\mathscr{L}^n$ to a limiting generator $\mathscr{L}$, in the sense that for any f "in a large class of functions" we have that $\mathscr{L}^n f(x) = \mathscr{L}f(x)$ uniformly in $x \in \mathbb{R}^n$. Assume also that the family $(\mathbb{P}^n)_n$ is tight on $\mathscr{C}$, then one can show that any accumulation point of $(\mathbb{P}^n)_n$ wrt. to the weak topology of probability measures is a solution of the martingale problem for $\mathscr{L}$.

c) (Markov process theory) If one can construct the semigroup $(P_t)_{t\geqslant 0}$ in the space of continuous functions $C(\mathbb{R}^n)$, associated to the operator $\mathscr{L}$ in the sense that $\partial_t P_t = \mathscr{L}P_t$ in the sense of Hille–Yoshida theory. Then one can construct a measure $\mathbb{P}$ using P to specify the finite dimensional distributions and then prove that it is a solution of the martingale problem. (this is stated here very vaguely).

Theorem 30. *(Stroock–Varadhan) Assume b, σ is are bounded measurable functions and a is bounded from below away from zero (in the sense of symmetric matrices) then there exists a solution to the martingale problem for $\mathscr{L}$ and the martingale problem for $\mathscr{L}$ has a unique solution.*

The condition on a means that there exists $\lambda > 0$ such that $\langle v, a(x) v\rangle_{\mathbb{R}^n} \geqslant \lambda \|v\|_{\mathbb{R}^n}^2$ for any $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ (ellipticity condition).

There is no further regularity requirement on the coefficients, i.e. they can be discontinuous.

Theorem 31. *(Skhorohod) Assume b, σ is are bounded measurable functions then there exists a weak solution of the SDE (equivalently, a solution of the martingale problem for $\mathscr{L}$).*

Theorem 32. *(Stroock–Varadhan) Assume b is a bounded measurable function, σ is continuous and $a = \sigma \sigma^T$ is bounded from below away from zero (in the sense of symmetric matrices) then the martingale problem for $\mathscr{L}$ has a unique solution.*

Remark 33. Most part of the theory exposed so far (e.g. pathwise uniqueness under Lipshitz conditions, Yamada–Watanabe theorem, Cherny's theorem, characterisation of martingale solutions/weak SDE) hold under the more general assumption that the coefficients of the SDE b, σ are adapted function of the "full history" of the process X, in the sense that

$$b: \mathbb{R}_+ \times \mathscr{C}^n \to \mathbb{R}^n, \qquad \sigma: \mathbb{R}_+ \times \mathscr{C}^n \to \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n),$$

such that the processes $(b(t, X))_{t \geqslant 0}$, $(\sigma(t, X))_{t \geqslant 0}$ are adapted to the filtration generated by X. Of course the Lipschitz condition on b, σ has to be read in the sense of the Banach space $C([0, t]; \mathbb{R}^n)$, i.e.

$$|b(t, f) - b(t, g)| + |\sigma(t, f) - \sigma(t, g)| \leqslant C_{b,\sigma} \|f - g\|_{C([0,t];\mathbb{R}^n)}$$

for all $t \geqslant 0$ and all $f, g \in \mathscr{C}^n = C(\mathbb{R}_+; \mathbb{R}^n)$. However the solutions of the SDE are not anymore in general Markov processes. Sometimes the SDE is called Markovian if the coefficients depends only on the current state, i.e. $b(t, f) = b(t, f(t))$, $\sigma(t, f) = \sigma(t, f(t))$.

6 Sufficient conditions for existence and uniqueness of SDEs

The more general result on uniqueness is the one for Lipschitz path–dependent coefficients:

Theorem 34. (Itô) *Assume that there exists a constant C such that for all $t \geqslant 0$ and $x, y \in C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D)$ we have*

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leqslant C \|x - y\|_{\infty,[0,t]},$$

$$|b_t(x)| + |\sigma_t(x)| \leqslant C(1 + \|x\|_{\infty,[0,t]}).$$

Then strong existence and pathwise uniqueness holds.

For a proof see [3].

In what follows we will restrict our considerations to coefficients which depends only on the present, namely

$$b_t(x) = b(t, x_t) \quad \text{and} \quad \sigma_t(x) = \sigma(t, x_t), \qquad t \geqslant 0, x \in C(\mathbb{R}_{\geqslant 0}; \mathbb{R}^D).$$

where $b: \mathbb{R}_{\geqslant 0} \times \mathbb{R}^D \to \mathbb{R}^D$ and $\sigma: \mathbb{R}_{\geqslant 0} \times \mathbb{R}^D \to \mathbb{R}^{D \times M}$ are measurable functions.

In one dimension we can relax the Lipschitz assumption on the diffusion coefficient up to a condition of Hölder regularity of $1/2$:

Theorem 35. (Yamada–Watanabe) *Assume $D = 1$, $b_t(x) = b(t, x_t)$ and $\sigma_t(x) = \sigma(t, x_t)$ and that there exists $C, \gamma > 0$ and an increasing function $h: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that*

$$\int_0^{\gamma} \frac{\mathrm{d}s}{h^2(s)} = +\infty$$

and

$$|b(t,x) - b(t,y)| \leqslant C|x-y|, \quad |\sigma(t,x) - \sigma(t,y)| \leqslant h(|x-y|), \qquad t \geqslant 0, \quad x, y \in \mathbb{R};$$

then pathwise uniqueness holds for (5).

For a proof see [4, Ch. V, Th. 40.1]. Weak existence can be established for continuous coefficients :

Theorem 36. (Skorokhod) *Assume that b, σ are continuous and bounded. Then there exists a weak solutions of (1).*

For a proof see [4, Ch. V, Th. 23.5].

Other results on existence/uniqueness are available, see [2].

Theorem 37. (Stroock–Varadhan) *Let $D = M$ and assume that b is measurable and bounded and that σ is continuous, bounded and such that for all $t \geqslant 0$ and $x \in \mathbb{R}^D$ there exists a constant $\varepsilon(t,x) > 0$ such that*

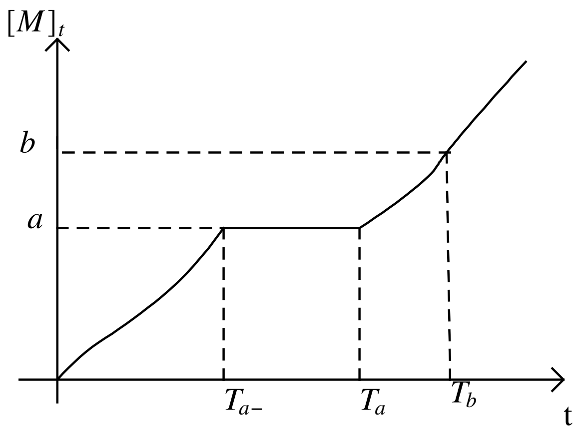
$$|\sigma(t,x) v| \geqslant \varepsilon(t,x)|v|, \qquad v \in \mathbb{R}^D.$$

Then there exists a weak solution and uniqueness in law holds.

7 Time change for continuous martingales and SDEs

Dambis, Dubins–Schwarz theorem says that "any continuous local martingale" is the time-change of a Brownian motion. Let us see how.

Let $(M_t)_{t\geqslant 0}$ cont. loc. mart. and consider its quadratic variation $([M]_t)_{t\geqslant 0}$, this is a continuous non-decreasing process. I'm looking for a time-change which "straighten out" the quadratic variation of M.



We define

$$T_a = \inf\{t \geqslant 0 : [M]_t > a\}$$

which is the right inverse of $[M]$, in the sense that

$$[M]_{T_a} = a$$

by continuity of $[M]$. However is only right-continuous, and if $t \in [T_{a-}, T_a]$ then $M_t = a$. $T_{a-} := \lim_{\varepsilon\downarrow 0} T_{a-\varepsilon}$.

In order for T_a to be a stopping time we need that $(\mathcal{F}_t)_{t\geqslant 0}$ is right continuous, this will be one of our assumptions. We also need that $[M]_\infty = +\infty$ (this assumption can be removed).

We use now $a \mapsto T_a$ as time change. Define a new filtration $(\mathcal{G}_a)_{a\geqslant 0}$ as $\mathcal{G}_a = \mathcal{F}_{T_a}$.

Now let

$$B_a := M_{T_a},$$

We want to prove that B is a Brownian motion via Levy's characterisation.

Let's prove continuity. It is clear that B_a is right-continuous since M is continuous and $T_\cdot$ is right-continuous. However $B_{a-} = \lim_{\varepsilon\downarrow 0} B_{a-\varepsilon} = M_{T_{a-}}$ and one observe that

$$\mathbb{P}(\exists a : M_{T_{a-}} \neq M_{T_a}) = 0.$$

The point here is that if $U < V$ are two stopping times and $[M]_U = [M]_V$ then $M_U = M_V$ a.s. Now we have $[M]_{T_{a-}} = a = [M]_{T_a}$ and $T_{a-} < T_a$ are stopping times, so

$$B_{a-} = M_{T_{a-}} = M_{T_a} = B_a$$

a.s. So $(B_a)_{a\geqslant 0}$ is continuous.

Let's prove that is a local martingale. We have $B_a = M_{T_a} = M_\infty^{T_a}$ recall that $M_t^T = M_{T\wedge t}$ is the stopped process. We have $[M^{T_a}]_\infty = [M]_{T_a} = a$ so $\mathbb{E}[B_a^2] = \mathbb{E}[[M^{T_a}]_\infty] = a$. By optional stopping we have for any $a > b$

$$\mathbb{E}[B_a|\mathscr{G}_b] = \mathbb{E}[M_\infty^{T_a}|\mathscr{F}_{T_b}] = M_{T_b} = B_b \qquad a.s.$$

so $(B_a)_{a\geqslant 0}$ is a local martingale.

We need now to compute its quadratic variation.

$$\mathbb{E}[B_a^2 - B_b^2|\mathscr{G}_b] = \mathbb{E}[(M_\infty^{T_a})^2 - (M_\infty^{T_b})^2|\mathscr{F}_{T_b}] = [M]_{T_a} - [M]_{T_b} = a - b$$

so the process $B_a^2 - a$ is a local martigale therefore $\langle B\rangle_a = a = [B]_a$.

And by Levy's theorem we conclude that $(B_a)_{a\geqslant 0}$ is a Brownian motion. Moreover

$$B_{[M]_t} = M_{T_{[M]_t}} = M_t \qquad a.s.$$

indeed $T_{[M]_t} \geqslant t$ without equality in general, but $[M]_{T_{[M]_t}} = [M]_t$ so $M_{T_{[M]_t}} = M_t$.

Theorem 38. *(Dambis, Dubins–Schwarz Brownian motion) Let $(M)_{t\geqslant 0}$ be a continuous local martingale for a right-continuous filtration $(\mathscr{F}_t)_{t\geqslant 0}$ then there exists a Brownian motion $(B_a)_{a\geqslant 0}$ (maybe in an enlarged probability space) such that $M_t = B_{[M]_t}$.*

Remark 39. If $[M]_\infty = +\infty$ then there is no need to enlarge the probability space.

Example 40. A first application: if $M_t = \int_0^t G_s \mathrm{d}W_s$ where W is a BM. Then $[M]_t = \int_0^t G_s^2 \mathrm{d}s$ and therefore

$$M_t = B_{\int_0^t G_s^2 \mathrm{d}s}$$

where B is the DDS BM of M.

Historically this has some interest because in the '40 Dœblin discovered Ito formula independently of Ito (but this was not known until 2000 because his results were in a letter which was sealed and to be opened not before that date). In Dœblin's approach there were no stochastic calculus, at the placve of stochastic integrals he was using time changed Brownian motions. His approach was limited to one dimension. For example, if W is a BM and $f \in C^2$ then

$$f(W_t) = f(W_0) + \frac{1}{2}\int_0^t f''(W_s)\mathrm{d}s + B_{\int_0^t f'(W_s)^2 \mathrm{d}s}$$

for some BM B.

If $(G_s)_{s\geqslant 0}$ is a deterministic function then this shows that $\left(\int_0^t G_s \mathrm{d}W_s\right)_{t\geqslant 0}$ is a Gaussian process.

Example 41. (Path-wise properties of martingales) Let B be the DDS BM of a local martingale M then we know that for any $\gamma < 1/2$ there exists a random constant C such that almost surely $C < \infty$ and

$$|B_a - B_b| \leqslant C|b-a|^\gamma, \qquad 0 \leqslant a, b \leqslant 1.$$

Therefore

$$|M_t - M_s| = |B_{[M]_t} - B_{[M]_s}| \leqslant C|[M]_t - [M]_s|^{\gamma}, \qquad 0 \leqslant t, s \leqslant T.$$

almost surely on the event $\{[M]_T \leqslant 1\}$.

Time-change of SDEs

Take the SDE (Y, W) where W is a BM and

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t,$$

where $\sigma(x) > 0$ for all $x \in \mathbb{R}$. Using the DDS BM B of Y we have

$$Y_t = B_{A_t}, \qquad A_t := [Y]_t = \int_0^t \sigma(Y_s)^2 \mathrm{d}s,$$

Let $(T_a)_{a \geqslant 0}$ be the inverse of $(A_t)_{t \geqslant 0}$, i.e. $A_{T_a} = a$ and $T_{A_t} = t$ (inverse exist since $(A_t)_{t \geqslant 0}$ is strictly increasing)

$$\mathrm{d}A_t = \sigma(Y_t)^2 \mathrm{d}t, \qquad \mathrm{d}t = \frac{\mathrm{d}A_t}{\sigma(Y_t)^2}$$

$$T_a = \int_0^a \frac{\mathrm{d}b}{\sigma(Y_{T_b})^2} = \int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2}$$

since $Y_{T_a} = B_a$. So aposteriori if B is a Brownian motion and I define

$$T_a = \int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2},$$

and I take A to be the inverse of T then $Y_t := B_{A_t}$ satisfies the SDE

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t$$

with some BM W. (This could be obtained also via the time-change in the martingale problem). Note that the time-change can be reconstructed back only knowing the trajectory of the BM. This can be used to find solutions to SDE.

Example 42. (Degenerate diffusion coefficient) Consider $\sigma(x) = |x|^{\alpha}$ for $x \in \mathbb{R}$ and $\alpha \in (0, 1/2)$. Let $(B_a)_{a \geqslant 0}$ be a 1d Brownian motion starting at $y \in \mathbb{R}$ and define

$$T_a := \int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2}.$$

In order to be sure that $(T_a)_{a \geqslant 0}$ is well defined we need only to be sure that $T_a < \infty$ a.s. Note that

$$\mathbb{E}[T_a] = \mathbb{E}\left[\int_0^a \frac{\mathrm{d}b}{\sigma(B_b)^2}\right] = \int_0^a \mathbb{E}\left[\frac{1}{|B_b|^{2\alpha}}\right]\mathrm{d}b = \int_0^a \left[\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{2b}}}{(2\pi b)^{1/2}} \frac{1}{|x|^{2\alpha}} \mathrm{d}x\right]\mathrm{d}b < \infty$$

by Fubini and using that $2\alpha < 1$ to have integrability near $x = 0$. We can define $t \mapsto A_t(\omega)$ to be the inverse of $a \mapsto T_a(\omega)$ (pathwise) and we can define $Y_t = B_{A_t}$. By our previous discussion this is a solution to the SDE

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t = |Y_t|^{\alpha}\mathrm{d}W_t$$

for some BM W. So we proved that this SDE has a non-trivial weak solution starting from any $Y_0 = y \in \mathbb{R}$, but now note that this SDE has also the solution $Y_t = 0$ if $y = 0$. So there is no uniqueness in law for this SDE. We have also pathwise non-uniqueness since the two weak solutions (Y, W) and $(0, W)$ do not coincide.

So in general we cannot expect uniqueness in law in one dimension if the diffusion coefficient is degenerate, i.e. $\sigma(x) = 0$ for some x.

Note that by the Yamada–Watanabe (existence) theorem as soon as $\sigma(x) = |x|^{\alpha}$ with $\alpha \geqslant 1/2$ we have pathwise uniqueness.

---

8 One dimensional (Markovian) diffusions

Let X be the solution of the SDE on the interval $(\alpha, \beta) \subset \mathbb{R}$

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$

where $b: (\alpha, \beta) \to \mathbb{R}$ and $\sigma: (\alpha, \beta) \to \mathbb{R}$ are continuous functions such that $\sigma(x) > 0$ for any $x \in (\alpha, \beta)$. The combination of time-change and space transformation allows a quite complete description of such kind of SDE.

Let's assume that $X_0 = x \in (\alpha, \beta)$ and that

$$\tau = \inf\{t \geqslant 0: X_t \notin (\alpha, \beta)\} = \sup_n \inf\{t \geqslant 0: X_t \in (\alpha + 1/n, \beta - 1/n)^c\}$$

the exit time of X from (α, β).

Coordinate transformation. Take a function $\varphi \in C^2((\alpha, \beta); \mathbb{R})$ and let $Y_t = \varphi(X_t)$. By appling Ito formula we have

$$\mathrm{d}Y_t = \sigma(X_t)\varphi'(X_t)\mathrm{d}B_t + \left(b(X_t)\varphi'(X_t) + \frac{1}{2}\sigma(X_t)^2\varphi''(X_t)\right)\mathrm{d}t, \qquad t \in [0, \tau].$$

Assume that $\varphi'(x) > 0$ so that φ si bijective onto its image and let φ^{-1} its inverse, then $X_t = \varphi^{-1}(Y_t)$ and if moreover φ satisfies

$$b(x)\varphi'(x) + \frac{1}{2}\sigma(x)^2\varphi''(x) = 0, \qquad x \in (\alpha, \beta) \tag{9}$$

then Y solves the SDE

$$\mathrm{d}Y_t = \tilde{\sigma}(Y_t)\mathrm{d}B_t, \qquad t \in [0, \tau],$$

with $\tilde{\sigma}(y) = (\sigma\varphi') \circ \varphi^{-1}(y) > 0$ and

$$\tau = \inf\{t \geqslant 0: X_t \notin (\alpha, \beta)\} = \inf\{t \geqslant 0: Y_t \notin (\varphi(\alpha), \varphi(\beta))\}.$$

Time change. Then Y is a local martingale (up to time τ) and if we let $A_t = [Y]_t = \int_0^t \tilde{\sigma}(Y_s)^2 \mathrm{d}s$ its quadratic variation, then we can define the time changed process $Z_a = Y_{T_a}$ with $T = A^{-1}$. We know that $(Z_a)_{a \geqslant 0}$ (maybe on a larger probability space) is a Brownian motion up to time σ given by $\sigma = A_\tau$ which corresponds to

$$\sigma = \inf\{a \geqslant 0: Z_a \notin (\varphi(\alpha), \varphi(\beta))\}$$

and that

$$Y_t = Z_{A_t}, \qquad T_a = \int_0^a \frac{\mathrm{d}b}{\tilde{\sigma}(Z_b)^2}, \qquad t \in [0, \tau], \quad a \in [0, \sigma].$$

So overall we can say that by "streching" space and time, any one dimensional diffusion is nothing more than a Brownian motion.

Exercise 1. Perform the coordinate transformation and the change of time on the martingale problem to arrive to the same conclusion.

On the other hand we can go back, i.e. start from the Brownian motion $(Z_a)_{a \geqslant 0}$ perform the time change T_a given above to obtain $Y_t = Z_{A_t}$ and then perform the coordinate transformation in the backward direction to obtain that $X_t = \varphi^{-1}(Y_t) = \varphi^{-1}(Z_{A_t})$ is the solution of the original SDE. So the original SDE has uniqueness in law because we have been able to express its law as a measurable transformation of the law of the Brownian motion. That is $X^\tau = \Phi(Z^\sigma)$ which implies $\mathrm{Law}(X^\tau) = \Phi_*(\mathrm{Law}(Z^\sigma))$.

Theorem 43. *Any solution X of any one dimensional SDE on (α, β) with $\sigma: (\alpha, \beta) \to \mathbb{R}_{>0}$ has the from $X_t = \varphi^{-1}(Z_{A_t})$ for $t \in [0, \tau]$ where τ is the exist time of X from (α, β), φ is the unique (up to shift and rescaling) solution of the ODE (9) such that $\varphi'(x) > 0$ for all $x \in (\alpha, \beta)$ and $A = T^{-1}$ with*

$$T_a = \int_0^a \frac{\mathrm{d}b}{\tilde{\sigma}(Z_b)^2}.$$

In particular such SDE has uniqueness in law.

Let us justify the fact that the ODE (9) has a unique solution with positive derivative (up to shift and rescaling). Note that φ has to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi'(x) = -2\frac{b(x)}{\sigma(x)^2}\varphi'(x), \qquad x \in (\alpha, \beta).$$

This is an ODE for $\varphi'(x)$ which is solved by

$$\varphi'(x) = B\exp\left(-2\int_{x_0}^x \frac{b(z)}{\sigma(z)^2}\mathrm{d}z\right), \qquad x \in (\alpha, \beta).$$

for some $B > 0$, and therefore

$$\varphi(x) = A + B\int_{x_0}^x \exp\left(-2\int_{x_0}^y \frac{b(z)}{\sigma(z)^2}\mathrm{d}z\right)\mathrm{d}y, \qquad x \in (\alpha, \beta).$$

Note that the non-uniqueness (up to shift and rescaling) of the ODE does not affect the conclusions of the theorem.

Remark 44. We would be interested to extend this result to non-continuous coefficients, however we loose the property that $\varphi \in C^2$ and we cannot use anymore Ito formula. (We might come back to this isssue when we discuss Tanaka's formula, which is an extension of Ito formula)

Remark 45. The non-degeneracy condition $\sigma(x) > 0$ is necessary, indeed we have seen that if $\sigma(x) = |x|^\rho$ with $\rho \in (0, 1/2)$ and $b(x) = 0$ then the SDE has no uniqueness in law.

Remark 46. Note that the function φ does not depen on α, β.

Note that the theorem says that for any one dimensional diffusion we have that if $\mathbb{P}$ is the law of a weak solution to the SDE starting in $x \in (\alpha, \beta)$, then if we let $(Z_a)_{a \geqslant 0}$ the associated BM we have that $Z_0 = \varphi(X_0) = \varphi(x) \in (\varphi(\alpha), \varphi(\beta))$ and if we assume that $\tau < \infty$ then $\sigma < \infty$ and

$$\mathbb{P}(X_\tau = \alpha) = \tilde{\mathbb{P}}(Z_\sigma = \varphi(\alpha)) = \frac{\varphi(\beta) - \varphi(x)}{\varphi(\beta) - \varphi(\alpha)}, \qquad x \in (\alpha, \beta).$$

Note also that the non-unqueness of φ does not affect this conclusion (as it should not!).

This kind of arguments can be used to study the recurrence or transience of more general processes. For example in a future exercise sheet we will see how to apply this to study recurrence of multidimensional Brownian motion and in general of Bessel processes.

Exercise 2. Using the coordinate transformation prove that the SDE has pathwise uniqueness when b is only continuous,

$$|\sigma(x) - \sigma(y)| \leqslant C|x - y|^{1/2}, \qquad x, y \in \mathbb{R}$$

and $\sigma(x) > 0$ for any $x \in \mathbb{R}$.

Bibliography

[1] A. Cherny. On the Uniqueness in Law and the Pathwise Uniqueness for Stochastic Differential Equations. *Theory of Probability & Its Applications*, 46(3):406–419, jan 2002.

[2] Alexander S. Cherny and Hans-Jürgen Engelbert. *Singular Stochastic Differential Equations*. Springer, Berlin, 2005 edition edition, jan 2005.

[3] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*. Springer, 3rd edition, dec 2004.

[4] L. C. G. Rogers and David Williams. *Diffusions, Markov Processes and Martingales: Volume 2, Itô Calculus*. Cambridge University Press, Cambridge, U.K. ; New York, 2 edition edition, sep 2000.

[5] Toshio Yamada and Shinzo Watanabe. On the uniqueness of solutions of stochastic differential equations. *Journal of Mathematics of Kyoto University*, 11(1):155–167, 1971.