V4F1 Stochastic Analysis – Problem Sheet 10

Tutorial classes: Wed July 6th 8–10 Chunqiu Song | Wed July 6th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday June 30th. At most in groups of two.

Let $(\Omega := C(\mathbb{R}_{\geqslant 0}; \mathbb{R}), \mathscr{F}, \mathscr{F}_\bullet, \mathbb{P})$ the one dimensional Wiener space and X the canonical process.

Exercise 1. [Pts 2+2+2+2+2] Find a predictable process F such that

$$\Phi = \mathbb{E}[\Phi] + \int_0^\infty F_s \mathrm{d}X_s$$

when $\Phi \in L^2(\Omega, \mathscr{F}_T, \mathbb{P})$ is each of the following r.v. (with $T > 0$ fixed)

$$X_T^2, \quad e^{X_T}, \quad \int_0^T X_t \mathrm{d}t, \quad X_T^3, \quad \sin(X_T).$$

(One possible approach: for any Φ try to find a martingale $(M_t)_t$ such that $M_T = \Phi$, and then apply Ito formula).

Exercise 2. [Pts 2+2+2] We want to prove that the linear span of r.v. of the form

$$E(h) = \cos\left(\int h_s \mathrm{d}X_s\right)\exp\left(\frac{1}{2}\int h_s^2 \mathrm{d}s\right), \quad F(h) = \sin\left(\int h_s \mathrm{d}X_s\right)\exp\left(\frac{1}{2}\int h_s^2 \mathrm{d}s\right), \qquad h \in L^2(\mathbb{R}_{\geqslant 0}),$$

is dense in $L^2(\Omega, \mathscr{F}, \mathbb{P})$ (h is a deterministic function and the integrals are over $\mathbb{R}_{\geqslant 0}$).

a) Show that if $G \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ is orthogonal to all $\{E(h), F(h) : h \in L^2(\mathbb{R})\}$, then in particular

$$\mathbb{E}[G \exp(i\lambda_1 B_{t_1} + \cdots + i\lambda_n B_{t_n})] = 0$$

for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $t_1, \cdots, t_n \geqslant 0$.

b) Deduce from this that G is orthogonal to all functions of the from $\phi(B_{t_1}, \ldots, B_{t_n})$ with $\phi \in C_0^\infty$. [Hint: use Fourier transform]

c) Conclude.

Exercise 3. [Pts 4+4] Use the class of functions introduced in Exercise 2 to reprove the Brownian martingale representation theorem.

a) Determine the martingale representation for functions Φ of the from

$$\Phi = \sum_i (a_i E(h_i) + b_i F(h_i))$$

where $a_i, b_i \in \mathbb{R}$, $h_i \in L^2(\mathbb{R}_{\geqslant 0})$ and the sum is finite.

b) Use the density of such functions to approximate an arbitrary element $\Phi \in L^2$ and conclude.