V4F1 Stochastic Analysis – Problem Sheet 2

Tutorial classes: Wed April 27th 8–10 Chunqiu Song | Wed April 27th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday April 21st. At most in groups of two.

Exercise 1. [Pts 2+3+2] Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion.

a) Define the process

$$X_t = a(t)\left(x_0 + \int_0^t b(s)\mathrm{d}B_s\right)$$

where $a, b\colon \mathbb{R}_+ \to \mathbb{R}$ are differentiable functions with $a(0) = 1$ and $a(t) > 0$. Compute the SDE satisfied by this process.

b) Use (a) to find an explicit solution for the SDEs in eqns.(1),(2),(3):

$$\begin{cases} \mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}B_t & t \in [0, T] \\ X_0 = x_0 \end{cases} \tag{1}$$

where α, σ, T are positive constants.

$$\begin{cases} \mathrm{d}X_t = -\frac{X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t & t \in [0, 1) \\ X_0 = 0 \end{cases} \tag{2}$$

$$\begin{cases} \mathrm{d}X_t = tX_t \mathrm{d}t + e^{t^2/2}\mathrm{d}B_t & t \in [0, T] \\ X_0 = 1 \end{cases} \tag{3}$$

c) Are the solutions of the SDEs in (b) strong and pathwise unique?

Exercise 2. [Pts 2+2+2] Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion.

a) Given $f \in C(\mathbb{R}_+)$, prove that $X_t = \int_0^t f(s)\mathrm{d}B_s$ is a Gaussian random variable with mean 0 and variance $\int_0^t f(u)^2 \mathrm{d}u$ for all $t \geqslant 0$.

b) The Ornstein–Uhlenbeck process $(X_t)_{t\geqslant 0}$ is defined as the solution to the SDE

$$\begin{cases} \mathrm{d}X_t = (-\alpha X_t + \beta)\mathrm{d}t + \sigma \mathrm{d}B_t & t \geqslant 0 \\ X_0 = x_0 \end{cases} \tag{4}$$

where α, σ are positive constant and $\beta, x_0 \in \mathbb{R}$. Find the explicit solution to the SDE (4).

c) Prove that X_t converges in distribution as $t \to \infty$ to a Gaussian random variable with mean β / α and variance $\sigma^2 / 2\alpha$.

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Exercise 3. [Pts 3+2+2] Let $(B_t)_{t\geqslant 0}$ be a 2-dimensional Brownian motion and X a two-dimensional stochastic process solution to the SDE

$$\begin{cases} \mathrm{d}X_t = AX_t\mathrm{d}t + \mathrm{d}B_t & t \geqslant 0 \\ X_0 = \xi \end{cases} \tag{5}$$

where ξ is a random variable in $\mathbb{R}^2$ independent of B and

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

with $\alpha \in \mathbb{R}$.

a) Let $\phi(t)$ be a 2×2 matrix that satisfies the ODE

$$\dot{\phi}(t) = A\phi(t), \qquad \phi(0) = \mathbb{I}_2$$

where $\mathbb{I}_2$ is the 2×2 identity matrix. Show that $\phi(t) = e^{At} = \sum_{n\geqslant 0} A^n \frac{t^n}{n!}$ and calculate $\phi(t)$ explicitly. Find $\phi(t)^{-1}$ (inverse matrix).

b) Verify that

$$X_t = \phi(t)\left(\xi + \int_0^t \phi(s)^{-1}\mathrm{d}B_s\right)$$

solves the SDE (5).

c) Calculate the explicit solution of (5).