V4F1 Stochastic Analysis – Problem Sheet 3

Tutorial classes: Wed May 4th 8–10 Chunqiu Song | Wed May 4th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday April 28th. At most in groups of two.

Exercise 1. [Pts 3] **(Constant quadratic variation)** Let M be a continuous local martingale and $S \leqslant T$ two stopping times. Prove that $[M]_T = [M]_S < \infty$ a.s implies $M_t = M_S$ for all $t \in [S, T]$ a.s. . *[Hint: consider the continuous local martingale $N_t = \int_0^t \mathbb{I}_{]S,T]}(s) \mathrm{d}M_s$].*

Exercise 2. [Pts 3+3]**(Feynman–Kac formula for Ito diffusions)**

a) Consider the solution X of the SDE in $\mathbb{R}^n$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \qquad X_0 = x,$$

where B is a d-dimensional Brownian motion and $b\colon \mathbb{R}^n \to \mathbb{R}^n$, $\sigma\colon \mathbb{R}^n \to \mathbb{R}^{n\times d}$ locally bounded continuous coefficients. Let $\mathscr{L}$ be the associated infinitesimal generator. Fix $t > 0$ and assume that $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ and $V\colon [0, t] \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ are continuous functions. Show that any bounded $C^{1,2}$ solution $u\colon [0, t] \times \mathbb{R}^n \to \mathbb{R}$ of the equation

$$\begin{aligned} \frac{\partial}{\partial s} u(s,x) &= \mathscr{L}u(s,x) - V(s,x)u(s,x), \qquad (s,x) \in (0,t] \times \mathbb{R}^n, \\ u(0,x) &= \varphi(x), \end{aligned}$$

has the stochastic representation

$$u(t,x) = \mathbb{E}\left[\varphi(X_t)\exp\left(-\int_0^t V(t-s, X_s)\mathrm{d}s\right)\right].$$

In particular, there is at most only one solution of the PDE.
[Hint: show that $M_r = \exp(-\int_0^r V(t-s, X_s)\mathrm{d}s)u(t-r, X_r)$ is a local martingale].

b) The price of a security is modeled by a geometric Brownian motion X with parameters $\alpha, \sigma > 0$:

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t, \qquad X_0 = x > 0.$$

At price y we have a running cost of $V(y)$ per unit time. The total cost up to time t is then

$$A_t = \int_0^t V(X_s)\mathrm{d}s.$$

Suppose that u is a bounded solution to the PDE

$$\begin{aligned} \frac{\partial}{\partial s} u(s,x) &= \mathscr{L}u(s,x) - \beta V(x)u(s,x), \qquad (s,x) \in (0,t] \times \mathbb{R}_{\geqslant 0}, \\ u(0,x) &= 1, \end{aligned}$$

where $\mathscr{L}$ is the generator of X. Show that the Laplace transform of A_t is given by

$$\mathbb{E}[e^{-\beta A_t}] = u(t,x).$$

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Exercise 3. [Pts 3+3+3+2] **(Continuous Branching Process)** Consider a family of diffusions $(X_t(x))_{t>0,x>0}$ satisfying the SDE

$$\mathrm{d}X_t(x) = \alpha X_t(x)\mathrm{d}t + \sqrt{\beta X_t(x)}\,\mathrm{d}B_t, \qquad X_0(x) = x,$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_{>0}$. Existence of strong solutions to this equation follows from the Yamada–Watanabe theorem. Let $(\tilde{X}, \tilde{B})$ be an independent copy of (X, B) and let $Y_t(x,y) = X_t(x) + \tilde{X}_t(y)$ for $t > 0, x > 0, y > 0$.

a) *(Branching)* Compute the SDE satisfied by Y and prove that $(Y(x,y))_{t \geqslant 0}$ has the same law of $(X_t(x+y))_{t \geqslant 0}$. *[Hint: use martingale caracterization of weak solutions and pathwise uniqueness]*

b) *(Duality)* Show that this implies that there exists a function $u\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ such that

$$\mathbb{E}[e^{-\lambda X_t(x)}] = e^{-xu(t,\lambda)}, \qquad x \in \mathbb{R}_{>0} \tag{1}$$

if we assume that the map $x \mapsto \mathbb{E}[e^{-\lambda X_t(x)}]$ is continuous.

c) Assume that $u\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ is differentiable with respect to its first parameter. Apply Ito formula to $s \mapsto G_s = e^{-u(t-s,\lambda)X_s(x)}$ and determine which differential equation u should satisfy in order for G to be a local martingale. Prove that in this case eq. (1) is satisfied (in particular, if a solution of the equation exists then it is unique).

d) *(Extinction probability)* Find the explicit solution u for the differential equation and using eq. (1) prove that if $\alpha = 0$ then

$$\mathbb{P}(X_t(x) = 0) = e^{-2x/(\beta t)}, \qquad x, t > 0.$$