V4F1 Stochastic Analysis – Problem Sheet 5

Tutorial classes: Wed May 18th 8–10 Chunqiu Song | Wed May 18th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday May 12th. At most in groups of two.

Exercise 1. [Pts 2+2+2] **(Passage time to a sloping line)** Let X be a one–dimensional Brownian motion with $X_0=0$ and let $a>0$, $b\in\mathbb{R}$.

a) Let $T_L=\inf\{t\geqslant 0: X_t=a+bt\}$ denote the first passage time to the line $y=a+bt$. Show that

$$\mathbb{P}(T_L\leqslant t)=\mathbb{E}[e^{-bX_t-b^2t/2}\mathbb{I}_{T_a\leqslant t}], \tag{1}$$

where $T_a=\inf\{t\geqslant 0: X_t=a\}$ is the first passage time to level a.

b) Recall that, by the reflection principle, the law of T_a is absolutely continuous with density

$$f_{T_a}(t)=a\,t^{-3/2}\varphi(a/\sqrt{t})\,\mathbb{I}_{(0,\infty)}(t),$$

where φ is the standard normal density. Deduce that the law of T_L is absolutely continuous with density

$$f_{T_L}(t)=a\,t^{-3/2}\varphi((a+bt)/\sqrt{t})\,\mathbb{I}_{(0,\infty)}(t).$$

[Hint: in (1) take the conditional expectation w.r.t. $\mathscr{F}_{T_a}$].

c) Show that, for $b>0$,

$$\mathbb{E}\left[e^{-bX_t}\max_{s\leqslant t}(X_s)\right]\simeq\frac{e^{b^2t/2}}{2b},\quad\text{and}\quad\mathbb{E}\left[e^{bX_t}\max_{s\leqslant t}(X_s)\right]\simeq b\,t e^{b^2t/2},\qquad\text{as } t\to\infty.$$

Exercise 2. [Pts 2+2+3] **(Brownian Bridge)** Let X be a d–dimensional Brownian motion with $X_0=0$.

a) Show that, for any $y\in\mathbb{R}^d$, the process

$$X_t^y=X_t-t\,(X_1-y)\qquad t\in[0,1]$$

is independent of X_1.

b) Let μ_y denote the law of X^y on $C([0,1];\mathbb{R}^d)$. Show that $y\mapsto\mu_y$ is a regular version of the conditional distribution of X given $X_1=y$.

c) Compute the SDE satisfied by the canonical process Y under the probability measure μ_y on the space $C([0,1];\mathbb{R}^d)$. (Hint: use Doob's h-transform argument from the lectures)

Exercise 3. [Pts 3] Let M be a positive continuous supermartingale such that $\mathbb{E}[M_0]<\infty$. Let $M_\infty=\lim_{t\to\infty}M_t$. Show that if $\mathbb{E}[M_\infty]=\mathbb{E}[M_0]$ then M is a martingale and $\mathbb{E}[M_\infty|\mathscr{F}_t]=M_t$. *[Hint: prove that $\mathbb{E}[M_\infty|\mathscr{F}_t]\leqslant M_t$ and that $\mathbb{E}[M_t]=\mathbb{E}[M_0]$ and conclude.]*

Exercise 4. [Pts 4] Prove directly that the h-transform gives a transformation of martingale problems from the one with drift b and diffusion σ to another with same diffusion coefficient σ but different drift $\tilde{b}$.