Institute for Applied Mathematics – SS2022
Massimiliano Gubinelli · Paolo Rinaldi

V4F1 Stochastic Analysis – Problem Sheet 6

Tutorial classes: Wed May 25th 8–10 Chunqiu Song | Wed May 25th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday May 19th. At most in groups of two.

Exercise 1. [Pts 2+2+2+2]**(Brownian motion writes your name)** Prove that a Brownian motion in $\mathbb{R}^2$ will write your name (in cursive script, without dotted 'i's or crossed 't's). Let B be a two dimensional Brownian motion on $[0,1]$ and observe that $X_t^{(a,b)} = (b-a)^{1/2}(B_{a+(b-a)t} - B_a)$ for $t \in [0,1]$ has the same law as B. Let g: $[0,1] \to \mathbb{R}^2$ a smooth parametrization of your name. Let us agree that the Brownian motion $X^{(a,b)}$ spells your name (to precision $\varepsilon > 0$) if

$$\sup_{t \in (0,1)} |X_t^{(a,b)} - g(t)| \leqslant \varepsilon. \tag{1}$$

a) For $k \in \mathbb{N}$ let A_k be the event that (1) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Check that the events $(A_k)_{k \in \mathbb{N}}$ are independent and $\mathbb{P}(A_k) = \mathbb{P}(A_0)$ for all $k \geqslant 0$. Conclude that if $\mathbb{P}(A_0) > 0$ then infinitely many of the A_ks will occur almost surely.

b) Show that

$$\mathbb{P}\left[\sup_{t \in (0,1)} |B_t| \leqslant \varepsilon\right] > 0. \tag{2}$$

c) Using (2) and Girsanov's transform to show that $\mathbb{P}(A_0) > 0$ (Hint: construct a measure $\mathbb{Q}$ so that $B_t - g(t)$ is a Brownian motion)

d) Prove that a similar result holds for g only continuous.

Exercise 2. [Pts 3] Let $(X, \mathbb{P})$ be a solution of the martingale problem with drift b and diffusion σ. Generalise appropriately the Girsanov transform to construct a measure $\mathbb{Q}$ under which the process X solves a martingale problem with a different drift. For simplicity, assume that all the necessary integrability conditions are satisfied. (What takes the place of the Brownian motion?)

Exercise 3. [Pts 3+3+3] Given smooth, bounded functions $A: \mathbb{R}^d \to \mathbb{R}^d$, $V: \mathbb{R}^d \to \mathbb{R}$. Consider the operator $H(A)$ on $L^2(\mathbb{R}^d)$ given by

$$H(A) = -\frac{1}{2}|\nabla - iA(x)|^2 + V(x)$$

We will assume that this operator is self–adjoint (with suitable domain), bounded from below and with discrete spectrum. We will denote $E_0(A)$ its smaller eigenvalue which we will assume simple (i.e. of multiplicity one). Let ψ the complex valued solution to

$$\partial_t \psi(t,x) = -H(A)\,\psi(t,x), \qquad \psi(0,x) = \psi_0(x),$$

which we will assume to exist, to be once differentiable in t and twice in x and be bounded with bounded derivatives.

a) Find a suitable functions $B, C: \mathbb{R}^d \to \mathbb{C}$ with which we can give the following Feynman–Kac representation for ψ:

$$\psi(t,x) = \mathbb{E}_x\left\{\psi_0(X_t)\exp\left[\int_0^t B(X_s)\mathrm{d}X_s + \int_0^t C(X_s)\mathrm{d}s\right]\right\}$$

where under $\mathbb{E}_x$ the process X is a d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$.

b) Prove that the lowest eigenvector of H_A is strictly positive everywhere.

c) Use the above representation to prove the *diamagnetic inequality*

$$E_0(A) \geqslant E_0(0).$$

[Hint: take $\psi_0(x) = 1$ *and argue that* $\psi(t,x) \simeq c e^{-E_0 t} \varphi(x) + o_t(1)$ *where* $H\varphi = E_0(A)\varphi$ *and conclude]*