V4F1 Stochastic Analysis – Problem Sheet 7

Tutorial classes: Wed June 1st 8–10 Chunqiu Song | Wed June 1st 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday May 26th. At most in groups of two.

Exercise 1. [Pts 3+3+3+4] Let X a solution of the SDE in $\mathbb{R}^n$

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t, \tag{1}$$

with a vectorfield $b: \mathbb{R}^n \to \mathbb{R}^n$ measurable and with linear growth.

a) Prove that for all $T > 0$, almost surely

$$A(T) = \int_0^T |b(X_s)|^2 \mathrm{d}s < \infty,$$

and therefore the process is unique in law.

b) Find a (deterministic) increasing function $f: \mathbb{R}_+ \to \mathbb{R}_+$ such that, almost surely

$$\sup_{T \geqslant 0} \frac{A(T)}{f(T)} < \infty.$$

[Hint: find a constant C such that $\sup_{T \geqslant 0} \frac{A(T)}{f(T)} \leqslant \sum_{n \geqslant 0} \frac{CA(n)}{f(n)} < \infty$ a.s.]

c) Use Girsanov's transform to prove that the process is Markov when b is a bounded vectorfield.

d) (Bonus) Try to extend the proof of the Markov property for b of linear growth.

Exercise 2. [Pts 5] Let $\mathscr{C}^n = C(\mathbb{R}_+, \mathbb{R}^n)$ with the Borel σ-field and $\mathbb{W}_x$ the law of the Brownian motion starting at x. Let X the unique solution of the SDE (1) with $b = -\nabla V$ and V a positive C^2 function such that

$$|\nabla V(x)|^2 - \Delta V(x) \geqslant -L \qquad x \in \mathbb{R}^n.$$

Use the path-integral formula

$$\mathbb{E}_x(f(X_T)) = \int_{\mathscr{C}^n} f(\omega_T) \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2}\int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s)) \mathrm{d}s \right) \mathbb{W}_x(\mathrm{d}\omega)$$

to show that for any two bounded functions f, g and under appropriate conditions on V:

$$\int (P_T f)(x) g(x) e^{-2V(x)} \mathrm{d}x = \int f(x) (P_T g)(x) e^{-2V(x)} \mathrm{d}x$$

which shows that P_T is symmetric wrt. the measure $e^{-2V(x)}\mathrm{d}x$ and taking $g = 1$ show that $e^{-2V(x)}\mathrm{d}x$ properly normalized is an invariant measure for the SDE

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \mathrm{d}B_t,$$

meaning that if X_0 is taken with probability distribution $\propto e^{-2V(x)}\mathrm{d}x$ then

$$\mathbb{E}[f(X_0)] = \mathbb{E}[f(X_T)],$$

for all $T \geqslant 0$.
[Hint: let $\mathbb{W}_{x,y}$ the conditional law of the Brownian motion ω to have $\omega_T = y$, i.e. the Brownian bridge. Prove that the under $\mathbb{W}_{x,y}$ the process $\tilde{\omega}_t = \omega_{T-t}$ has law $\mathbb{W}_{y,x}$ and use the path integral]

Exercise 3. [Pts 3+3] Prove a Fubini theorem for stochastic integrals. Let $(\Lambda, \mathscr{A})$ be a measure space and $(\Omega, \mathscr{F}, \mathscr{F}_\bullet, \mathbb{P})$ a filtered probability space.

a) Let $(X_n)_n$ a sequence of functions $X_n: \Omega \times \Lambda \to \mathbb{R}$ which are $\mathscr{F} \otimes \mathscr{A}$ measurable (product σ-field) and such that $(X_n(\cdot, \lambda))_n$ converges in probability for any fixed $\lambda \in \Lambda$. Prove that there exists an $\mathscr{F} \otimes \mathscr{A}$ measurable function $X: \Omega \times \Lambda \to \mathbb{R}$ for which $X_n(\cdot, \lambda) \xrightarrow[\mathbb{P}]{} X(\cdot, \lambda)$ for any $\lambda \in \Lambda$. [Hint: here the difficulty is the measurability of the limit X, consider the sequence $n_k(\lambda)$ defined by $n_0(\lambda) = 1$ and

$$n_{k+1}(\lambda) = \inf\left\{ m > n_k(\lambda) \colon \sup_{p,q \geqslant m} \mathbb{P}[|X_p(\cdot, \lambda) - X_q(\cdot, \lambda)| > 2^{-k}] \leqslant 2^{-k} \right\}$$

and prove that $\lim_k X_{n_k(\lambda)}(\cdot, \lambda)$ exists a.s. and conclude]

b) Let $H: \Lambda \times \mathbb{R}_{\geqslant 0} \times \Omega \to R$ be a bounded function which is measurable w.r.t. $\mathscr{A} \otimes \mathscr{P}$ where $\mathscr{P}$ is the predictable σ-field on $\mathbb{R}_{\geqslant 0} \times \Omega$. Let M be a continuous martingale on $(\Omega, \mathscr{F}, \mathscr{F}_\bullet, \mathbb{P})$. Prove that there exists a function $J: \Lambda \times \Omega \to \mathbb{R}$ measurable for $\mathscr{A} \otimes \mathscr{F}_T$ which is a version of the stochastic process $\lambda \mapsto J(\lambda) := \int_0^T H(\lambda, s) \mathrm{d}M_s$ and for which it holds

$$\int_\Lambda J(\lambda) m(\mathrm{d}\lambda) = \int_0^T \left[\int_\Lambda H(\lambda, s, \cdot)\, m(\mathrm{d}\lambda) \right] \mathrm{d}M_s, \qquad a.s.$$

for any bounded measure m on $(\Lambda, \mathscr{A})$. Hint: prove that

$$\mathbb{E}\left[\left(\int_0^T \left[\int_\Lambda H(\lambda, s, \cdot)\, m(\mathrm{d}\lambda) \right] \mathrm{d}M_s - \int_\Lambda J(\lambda) m(\mathrm{d}\lambda) \right)^2 \right] = 0.$$

[Taken from Revuz-Yor, Chap. 4]