Institute for Applied Mathematics – SS2022
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V4F1 Stochastic Analysis – Problem Sheet 8

Tutorial classes: Wed June 8th 8–10 Chunqiu Song | Wed June 8th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday June 2nd. At most in groups of two.

Exercise 1. [Pts 2+2+2+2] Assume that $\Omega = C(R_{\geqslant 0}; \mathbb{R}^d)$, $\mathbb{P}$ is the d–dimensional Wiener measure and that X is the canonical process on Ω and that the filtration $\mathscr{F}_\bullet$ is generated by X. Consider a predictable $\mathbb{R}^d$-valued drift b given by a function $b: \mathbb{R}_{\geqslant 0} \times \Omega \to \mathbb{R}^d$. By tilting $\mathbb{P}$ via $Z = \mathscr{E}(\int_0^\cdot b(X) \mathrm{d}X)$ we obtain that, under the tilted measure $\mathbb{P}^b$ the process X is a solution of the SDE

$$\mathrm{d}X_t = b_t(X) + \mathrm{d}W_t, \qquad t \geqslant 0$$

where W is a $\mathbb{P}^b$–Brownian motion.

a) Prove that if

$$|b_t(x)| \leqslant C(1 + |x_t|), \qquad t \geqslant 0, x \in \Omega,$$

then Novikov's condition holds conditionally on $\mathscr{F}_s$ for intervals $[s, t]$ such that $|t - s|$ is small enough, i.e.

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_s^t |b_u(X)|^2 \mathrm{d}u\right) | \mathscr{F}_s\right] < +\infty.$$

b) Deduce that Z is a martingale. [Hint: prove that $\mathbb{E}[Z_t | \mathscr{F}_s] = Z_s$ for small time intervals $[s, t]$ and the conclude].

c) Prove that

$$\mathbb{P}(\|X\|_{[0,t]} > r) \leqslant 2 d e^{-r^2/2dt} \qquad t \geqslant 0, r \geqslant 0.$$

where $\|X\|_{[0,t]}$ denotes the supremum wrt. the Euclidean norm of $(X_s)_{s \in [0,t]}$.
[Hint: use Doob's inequality for the submartingale $e^{\lambda X_t^i}$ and optimize over $\lambda > 0$]

d) Prove the same result as in (a) under the more general assumption that b is a previsible drift such that

$$|b_t(x)| \leqslant C(1 + \|x\|_{\infty,[0,t]}), \qquad t \geqslant 0, x \in \Omega$$

where $C < +\infty$.

Exercise 2. [Pts 2+2+2] Consider the one dimensional SDE

$$\mathrm{d}X_t = -X_t^3 \mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x,$$

where B is a standard Brownian motion.

a) Let $f(t, x) = (1 + |x|^2)$ and $T_L = \inf\{t \geqslant 0 : |X_t| > L\}$. Use Ito formula to show that there exists a constant λ such that the process $Z_t := e^{-\lambda(t \wedge T_L)} f(X_{t \wedge T_L})$ is a supermartingale.

b) Deduce that $\mathbb{P}(T_L \leqslant t) \to 0$ as $L \to \infty$.

c) Conclude that solutions of the SDE cannot explode (that is $\zeta := \sup_L T_L = \infty$ a.s.).

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Exercise 3. [Pts 2+2+2] If $c(t)=(x(t),y(t))$ is a smooth curve in $\mathbb{R}^2$ with c(0) = 0,

$$A_t=\int_0^t (x(s)y'(s)-y(s)x'(s))\mathrm{d}s$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0,t]$. Analogously, for a two-dimensional Brownian motion $B_t=(X_t,Y_t)$ with $B_0=0$, one defines the Lévy Area

$$A_t=\int_0^t (X_s\mathrm{d}Y_s-Y_s\mathrm{d}X_s).$$

a) Let $\alpha(t)$, $\beta(t)$ be C^1-functions, $p\in\mathbb{R}$, and

$$V_t=ipA_t-\frac{\alpha(t)}{2}(X_t^2+Y_t^2)+\beta(t).$$

Use Itô formula to show that e^{V_t} is a local martingale provided $\alpha'(t)=\alpha(t)^2-p^2$ and $\beta'(t)=\alpha(t)$

b) Let $t_0\geqslant 0$. Solutions to the equations for α,β with $\alpha(t_0)=\beta(t_0)=0$ are

$$\alpha(t)=p\tanh(p(t_0-t)),\qquad \beta(t)=-\log\cosh(p(t_0-t)).$$

Conclude that

$$\mathbb{E}[e^{ipA_{t_0}}]=(\cosh(pt_0))^{-1}.$$

c) Show that the distribution of A_t is absolutely continuous with respect to the Lebesgue measure with density

$$f_{A_t}(x)=(2t\cosh(\pi x/2t))^{-1},\qquad x\in\mathbb{R}.$$