Note 11

Brownian motion.

(Lectures and notes by Francesco De Vecchi)

1 Definition and equivalent characterizations

Definition 1. *A stochastic process $B_\cdot : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a Brownian motion if*

1. $B_0 = 0$,
2. *for any* $0 \le t_1 \le t_2 \le \ldots \le t_n \in \mathbb{R}_+$ *we have that* $B_{t_1} - B_0$, $B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent random variables and* $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$,
3. *for almost every* $\omega \in \Omega$ *the function* $t \longmapsto B_t(\omega)$ *is continuous (i.e., in* $C^0(\mathbb{R}_+, \mathbb{R})$*).*

1.1 Brownian motion as a Markov process

We consider the following completed natural filtration of B_t given by

$$F_t = \sigma(B_s, s \in [0, t]).$$

Theorem 2. *A Brownian motion B_t is a F_t Markov process with transition kernel given by*

$$p(x, t; y, s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right), \tag{1}$$

where $0 \le s < t$.

Proof. We have to prove that for any $0 \le s < t$ and any Borel set $A \subset \mathbb{R}$ there exists a version of $\mathbb{P}(B_t \in A | F_s)$ which is $\sigma(B_s)$ measurable.

By Definition 1 we have that $B_t - B_s$ is independent of $B_s - B_0 = B_s$ and $B_t - B_s \sim N(0, t-s)$

$$\begin{aligned}\mathbb{P}(B_t \in A | F_s) &= \mathbb{P}((B_t - B_s) + B_s \in A | F_s) \\ &= \int_A \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(\frac{-(x-B_s)^2}{2(t-s)}\right).\end{aligned}$$

□

Corollary 3. *For any $0 < t_1 < t_2 < \cdots < t_n$ we have that the law of $(B_{t_1}, \ldots, B_{t_n})$ is given by*

$$\frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \exp\left(-\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right), \tag{2}$$

where $t_0 = 0$ *and* $x_0 = 0$.

Proof. We prove the theorem for $n = 2$. The general case can be proved by induction.

Let A_1, A_2 be two Borel subsets of $\mathbb{R}$, then we have

$$\begin{aligned}
\mathbb{P}(B_{t_1} \in A_1, B_{t_2} \in A_2) &= \int_{A_1} \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x_1^2}{2t_1}\right) \mathbb{P}(B_{t_2} \in A_2 | B_{t_1} = x_1) \mathrm{d}x_1 \\
&= \int_{A_1} \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x_1^2}{2t_1}\right) \left(\int_{A_2} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left(\frac{-(x_2 - x_1)^2}{2(t_2 - t_1)}\right) \mathrm{d}x_2\right) \mathrm{d}x_1
\end{aligned}$$

where to obtain the last equality we use Theorem 2. □

Corollary 4. *Let* B_t *be a Markov process with transition kernel (1),* $B_0 = 0$ *and such that for almost every* $\omega \in \Omega$ *the function* $t \longmapsto B_t(\omega)$ *is in* $C^0(\mathbb{R}_+, \mathbb{R})$*, then* B_t *is a Brownian motion.*

Proof. We have only to prove that B_t satisfies the second property of Definition 1. Using the same reasoning of Corollary 3, we obtain that, if B_t is a Markov process with transition kernel (1), then it has finite dimensional marginals given by (2). This implies that for any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \in \mathbb{R}_+$ we have that $B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent random variables and $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$. □

1.2 Brownian motion as a Gaussian process

Theorem 5. *Brownian motion is a Gaussian process such that* $B_0 = 0$ *and*

$$\begin{aligned}
\mathbb{E}[B_t] &= 0 && (3) \\
\mathrm{cov}(B_t, B_s) &= \min(t, s). && (4)
\end{aligned}$$

Proof. The fact that Brownian motion is a Gaussian process follows by the explicit expression of finite dimensional marginals given in Corollary 3.

Using the definition of Brownian motion we have $\mathbb{E}[B_t] = \mathbb{E}[B_t - B_0] = 0$ and, if $s \leq t$,

$$\mathrm{cov}(B_t, B_s) = \mathrm{cov}(B_t - B_s, B_s) + \mathrm{cov}(B_s, B_s) = s.$$ □

Corollary 6. *Let* B_t *be a Gaussian process with mean (3) and co-variance (4), and suppose that* $B_0 = 0$ *and for almost every* $\omega \in \Omega$ *the function* $t \longmapsto B_t(\omega)$ *is in* $C^0(\mathbb{R}_+, \mathbb{R})$*, then* B_t *is a Brownian motion.*

Proof. We have only to prove that B_t satisfies the second property of Definition 1. Since $B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are Gaussian random variables (being linear combinations of jointly Gaussian random variables) we have to prove that $\mathrm{cov}(B_{t_i} - B_{t_{i-1}}, B_{t_j} - B_{t_{j-1}}) = 0$ if $i \neq j$. Suppose that $t_j < t_i$ then

$$\begin{aligned}
\mathrm{cov}(B_{t_i} - B_{t_{i-1}}, B_{t_j} - B_{t_{j-1}}) &= \mathrm{cov}(B_{t_i}, B_{t_j}) - \mathrm{cov}(B_{t_{i-1}}, B_{t_j}) - \mathrm{cov}(B_{t_i}, B_{t_{j-1}}) + \mathrm{cov}(B_{t_{i-1}}, B_{t_j}) \\
&= t_j - t_j - t_{j-1} + t_{j-1} = 0,
\end{aligned}$$

which conludes the proof. □

2 Lévy construction of Brownian motion

2.1 Haar and Schauder functions

We define Haar functions $h_n^k(t)$ for $n=0,1,\ldots\in\mathbb{N}$ and $k=0,\ldots,2^{n-1}-1$ in the following way: for $n=0$ we put $h_0^0(t)=1$ and for $n\neq 0$ we write

$$h_n^k(t)=2^{\frac{n-1}{2}}\left(\mathbb{I}_{\left[\frac{2k}{2^n},\frac{2k+1}{2^n}\right)}(t)-\mathbb{I}_{\left[\frac{2k+1}{2^n},\frac{2k+2}{2^n}\right)}(t)\right).$$

We define also Schauder functions as

$$e_n^k(t)=\int_0^s h_n^s(s)\mathrm{d}s.$$

Lemma 7. *The set of Haar functions forms an orthonormal basis of* $L^2([0,1])$.

Proof. The orthonormality is a consequence of the fact that $h_n^k(t)$ and $h_n^{k'}(t)$ are supported in different sets when $k\neq k'$, and that $h_n^k(t)$ has integral 0 on the dyadic set of the form $\left[\frac{k'}{2^{n-1}},\frac{k'+1}{2^{n-1}}\right]$ (for any $k'\in\mathbb{N}$).

In order to prove that the Haar functions form a complete basis of $L^2([0,1])$ we have only to prove that for any function $f\in L^2([0,1])$ such that $\int_0^1 f(t)h_n^k(t)=0$ we have $f=0$.

Consider the probability space $([0,1],B,\mathrm{d}x)$ (where B is the complete σ-algebra generated by Borel sets and dx is the Lebesgue measure) and consider the filtration $B_n=\left\{\left[\frac{k}{2^n},\frac{k+1}{2^n}\right],k=0,\ldots,2^n-1\right\}$, with $n\in\mathbb{N}$. It is clear that $\sigma(B_n|n\in\mathbb{N})=B$. If $\int_0^1 f(t)h_n^k(t)=0$ for $n\leq N$ then $\int_{\left[\frac{k}{2^n},\frac{k+1}{2^n}\right]} f(t)=0$ for $n\leq N$. This implies that

$$f_n=\mathbb{E}[f|B_n]=0.$$

On the other hand $\int_0^1 f_n^2(t)\mathrm{d}t=0$ and so f_n is a B_n martingale bounded in $L^2([0,1])$. Thus, by Doob Convergence Theorem for martingales, we have that $f_n\to\mathbb{E}[f|B]=f$ in $L^1([0,1])$. This implies that $f=\lim f_n=0$. □

Lemma 8. *We have that* $\sup_{t\in[0,1]}|e_n^k(t)|\leq 2^{-\frac{n-1}{2}}$ *and the series*

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^{n-1}-1}e_n^k(t)e_n^k(s)\right)=\min(t,s) \tag{5}$$

is absolutely convergent and it is equal to $\min(t,s)$.

Proof. The bound on $|e_n^k(t)|$ follows by a direct computation. In order to prove equality (5) we note that $\int_0^1 \mathbb{I}_{[0,t]}(\tau)h_n^k(\tau)\mathrm{d}\tau = e_n^k(t)$ (and a similar relation holds for $e_n^k(s)$). Using Parseval identity for orthonormal bases in an Hilbert space we obtain

$$\begin{aligned}\min(t,s) &= \int_0^1 \mathbb{I}_{[0,t]}(\tau)\mathbb{I}_{[0,s]}(\tau)\mathrm{d}\tau \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^{n-1}-1}\int_0^1 \mathbb{I}_{[0,t]}(\tau)h_n^k(\tau)\mathrm{d}\tau\int_0^1 \mathbb{I}_{[0,s]}(\tau)h_n^k(\tau)\mathrm{d}\tau\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^{n-1}-1} e_n^k(t)e_n^k(s)\right)\end{aligned}$$

and the previous series is absolutely convergent. □

2.2 Lévy construction of Brownian motion

Let $Z_{n,k}(\omega)$ be a sequence of independent random variables such that $Z_{n,k}\sim N(0,1)$. Consider the following sequence of stochastic processes

$$B_t^N(\omega) = \sum_{n=0}^{N}\left(\sum_{k=0}^{2^{n-1}-1} Z_{n,k}(\omega)e_n^k(t)\right).$$

From now on we restrict Definition 1, to processes of the form $B\colon[0,1]\times\Omega\to\mathbb{R}$, i.e., defined only on the set $[0,1]$ and not on the whole positive real line $\mathbb{R}_+$.

If we have a sequence of independent Brownian motions $\tilde{B}_t^1,\ldots,\tilde{B}_t^n$ defined on $[0,1]$, we can easily build a Brownian motion B_t defined on the whole real positive line $\mathbb{R}_+$ in the following way: if $n-1<t\leq n$ (where $n\in\mathbb{N}$) we define $B_t=\sum_{k=1}^{n-1}B_1^k+B_{t-n+1}^n$.

Theorem 9. *The sequence of stochastic processes B_t^N is almost surely convergent on $[0,1]$. Let B_t be the limit of B_t^N, then B_t is a Brownian motion on $[0,1]$.*

Proof. First we prove that the sequence of functions $t\longmapsto B_t^N(\omega)$ is uniformly convergent in $C^0([0,1],\mathbb{R})$ for almost every $\omega\in\Omega$. In order to prove this, we use Weierstrass criterion for uniform convergence in $C^0([0,1],\mathbb{R})$, proving that, writing $K_n(\omega)=\sup_{t\in[0,1]}\left|\sum_{k=0}^{2^{n-1}-1}Z_{n,k}(\omega)e_n^k(t)\right|$, we have $\sum_{n=0}^{\infty}K_n<+\infty$ almost surely.

Using the fact that for fixed n the functions $e_n^k(t)$ have disjoint support, and exploiting the bound $\sup_{t\in[0,1]}|e_n^k(t)|\leq 2^{-\frac{n-1}{2}}$, we have that

$$K_n(\omega)\leq 2^{-\frac{n-1}{2}}\sup_k|Z_{n,k}(\omega)|.$$

We want to prove that there exists a positive random variable $C\colon\Omega\to\mathbb{R}$, almost surely finite, such that

$$\sup_k|Z_{n,k}(\omega)|\leq nC(\omega).$$

Define $B_n=\{\omega|\sup_k|Z_{n,k}(\omega)|>n\}$ then $C(\omega)<+\infty$ whenever $\omega\notin\limsup_n B_n$. If we are able to prove that $\mathbb{P}(\limsup_n B_n)=0$ then $C(\omega)<+\infty$ almost surely. In order to prove that $\mathbb{P}(\limsup_n B_n)=0$, we use Borel-Cantelli Lemma and the fact that $\sum_n\mathbb{P}(B_n)<+\infty$.

Indeed

$$\mathbb{P}(B_n) \le \sum_{k=0}^{2^{n-1}-1} \mathbb{P}(|Z_{n,k}(\omega)| > n) \le \frac{2^n}{n\sqrt{2\pi}} \exp\left(\frac{-n^2}{2}\right)$$

where we used the fact that $Z_{n,k} \sim N(0,1)$. This implies that

$$\sum_n \mathbb{P}(B_n) \le \sum_n \frac{2^n}{n\sqrt{2\pi}} \exp\left(\frac{-n^2}{2}\right) < +\infty$$

which means that $C < +\infty$ almost surely. On the other hand we have that $K_n(\omega) \le 2^{-\frac{n-1}{2}} \sup_k |Z_{n,k}(\omega)|$ and so

$$\sum_n K_n(\omega) \le \sum_n 2^{-\frac{n-1}{2}} \sup_k |Z_{n,k}(\omega)| \le C(\omega) \sum_n n 2^{-\frac{n-1}{2}} < +\infty.$$

Thus the sequence $B_t^N(\omega)$ is almost surely convergent in $C^0([0,1],\mathbb{R})$.

Let B_t denote the limit of B_t^N when B_t^N is convergent and 0 otherwise. We have that B_t satisfies the condition 1 and 3 of Definition 1. In order to prove that B_t satisfies property 2 of Definition 1 we prove that B_t is a Gaussian process such that $\mathbb{E}[B_t] = 0$ and $\mathrm{cov}(B_t, B_s) = \min(s,t)$. Using Corollary 6, this is equivalent to prove that B_t is a Brownian motion.

First we prove that for any $t \in [0,1]$ the sequence of random variables B_t^N converges to B_t in $L^2(\Omega)$. Since B_t^N converges to B_t almost surely it is sufficient to prove that B_t^N forms a Cauchy sequence in $L^2(\Omega)$. We have that

$$\begin{aligned}
\mathbb{E}[(B_t^N - B_t^M)^2] &= \mathbb{E}\left[\left(\sum_{n=M}^{N} \left(\sum_{k=0}^{2^{n-1}-1} Z_{n,k}(\omega) e_n^k(t)\right)\right)^2\right] \\
&= \sum_{n=M}^{N} (e_n^k(t))^2
\end{aligned}$$

when $M \le N$ and using the fact that $Z_{n,k}$ are i.i.d. normal random variables with variance 1. On the other hand, by Lemma 8, the series $\sum_{n=0}^{+\infty} (e_n^k(t))^2 = t < +\infty$ is absolutely convergent, this means that

$$\lim_{M\to\infty} \sum_{n=M}^{N} (e_n^k(t))^2 = 0,$$

which implies that B_t^N is a Cauchy sequence in $L^2(\Omega)$.

The fact that $(B_{t_1}^N, \ldots, B_{t_n}^N)$ converges to $(B_{t_1}, \ldots, B_{t_n})$ in $L^2(\Omega)$ implies that B_t is a normal stochastic process (being the L^2 limit of a normal stochastic process), with $\mathbb{E}[B_t] = \lim_N \mathbb{E}[B_t^N]$ and $\mathrm{cov}(B_t, B_s) = \lim_N \mathrm{cov}(B_t^N, B_s^N)$. On the other hand we have that $\lim_N \mathbb{E}[B_t^N] = \lim_N 0 = 0$ and, by Lemma 8,

$$\lim_N \mathrm{cov}(B_t^N, B_s^N) = \lim_N \mathbb{E}[B_t^N B_s^N] = \lim_N \sum_{n=0}^{N} \left(\sum_{k=0}^{2^{n-1}-1} e_n^k(t) e_n^k(s)\right) = \min(t,s). \qquad \square$$

3 Regularity properties of Brownian motion

3.1 Non differentiability of Brownian motion

Let $M \subset \Omega$ be the measurable set

$$M = \{\omega \in \Omega, \text{there exists } \tau \in [0,1] \text{ such that } t \mapsto B_t(\omega) \text{ is differentiable in } \tau\}.$$

We want to prove that $\mathbb{P}(M) = 0$. This implies that the function $t \mapsto B_t(\omega)$ is everywhere non differentiable for almost every $\omega \in \Omega$.

Theorem 10. *Using the previous notation, if B_t is a Brownian motion then $\mathbb{P}(M) = 0$.*

Proof. We introduce the set

$$N = \left\{\omega \in \Omega, \text{there are } t \in [0,1] \text{ and } L, k \in \mathbb{N} \text{ such that } |B_t(\omega) - B_s(\omega)| < L|t-s| \text{ for any } s \in \left[t, t+\frac{1}{k}\right]\right\}.$$

Obviously $M \subset N$, so if we are able to prove that $\mathbb{P}(N) = 0$ we have proved $\mathbb{P}(M) = 0$.

If $n > 4k$ we can find $i \in \{1, 2, \ldots, n\}$ such that $\left(\frac{j}{n}, \frac{j+1}{n}\right) \subset \left[t, t+\frac{1}{k}\right]$ for $j = i, i+1, i+2$. If $\omega \in N$ we have

$$\left|B_{\frac{j}{n}}(\omega) - B_{\frac{j+1}{n}}(\omega)\right| \leq \left|B_{\frac{j}{n}}(\omega) - B_t(\omega)\right| + \left|B_t(\omega) - B_{\frac{j+1}{n}}(\omega)\right| \leq \frac{8L}{n}. \tag{6}$$

Let $\tilde{N}_{L,k}$ be the set defined as follows

$$\tilde{N}_{L,k} = \bigcap_{n>4k} \bigcup_{i=1}^{n} \left\{\left|B_{\frac{j}{n}}(\omega) - B_{\frac{j+1}{n}}(\omega)\right| \leq \frac{8L}{n}, \text{for } j = i, i+1, i+2\right\}.$$

We have that

$$\mathbb{P}\left(\left\{\left|B_{\frac{j}{n}}(\omega) - B_{\frac{j+1}{n}}(\omega)\right| \leq \frac{8L}{n}, \text{for } j = i, i+1, i+2\right\}\right) \leq \mathbb{P}\left(|Z| \leq \frac{8L}{\sqrt{n}}\right) \leq \frac{16^3 L^3}{n^{3/2}\sqrt{(2\pi)^3}}$$

where $Z \sim N(0,1)$. This means that

$$\mathbb{P}(\tilde{N}_{L,k}) \leq \inf_{n \in \mathbb{N}} n \cdot \mathbb{P}\left(\left\{\left|B_{\frac{j}{n}}(\omega) - B_{\frac{j+1}{n}}(\omega)\right| \leq \frac{8L}{n}, \text{for } j = i, i+1, i+2\right\}\right) \leq \inf_{n \in \mathbb{N}} \frac{16^3 L^3}{n^{1/2}\sqrt{(2\pi)^3}} = 0.$$

Since, by inequality (6), $N \subset \cup_{k,L \in \mathbb{N}} \tilde{N}_{L,k}$, we have that $\mathbb{P}(N) \leq \sum_{L,k \in \mathbb{N}} \mathbb{P}(\tilde{N}_{L,k}) = 0$. □

3.2 Hölder continuity of Brownian motion

Definition 11. *A function $f \in C^0([0,1], \mathbb{R})$ is called Hölder continuous of index $\alpha \in (0,1)$ if*

$$\sup_{0 \leq s < t \leq 1} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < +\infty.$$

In this case we use the notation $f \in C^{\alpha}([0,1],\mathbb{R})$.

Theorem 12. *If B_t is a Brownian motion we have that the function $t \mapsto B_t(\omega)$ is Hölder continuous of index α for any $\alpha \in \left(0,\frac{1}{2}\right)$ and for almost every $\omega \in \Omega$.*

Proof. From Theorem 9, we have that the series $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^{n-1}-1} Z_{n,k}(\omega)e_n^k(t)\right)$ is uniformly and absolutely convergent to the Brownian motion $B_t(\omega)$ almost surely. Let ω be an element of Ω for which the previous series is convergent, then we have that

$$|B_t(\omega)-B_s(\omega)| \leq \sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}-1} |Z_{n,k}(\omega)||e_n^k(t)-e_n^k(s)|.$$

Using the proof of Theorem 9, we have that there exists an almost surely finite and positive random variable $C(\omega)$ such that $\sup_k |Z_{n,k}(\omega)| \leq n \cdot C(\omega)$. Furthermore, by the definition of Schauder functions, we have that

$$\begin{aligned}
|e_n^k(t)-e_n^k(s)| &\leq 2^{\frac{n-1}{2}}|t-s| \\
|e_n^k(t)-e_n^k(s)| &\leq 2^{-\frac{n-3}{2}}.
\end{aligned}$$

Fix $0 \leq s < t \leq 1$ and let $N \in \mathbb{N}$ be such that

$$2^{-N} < |t-s| \leq 2^{-(N-1)}.$$

Using the fact that in the sum $\sum_{k=0}^{2^{n-1}-1} |e_n^k(t)-e_n^k(s)|$ only at most two addends are non zero, we obtain

$$|B_t(\omega)-B_s(\omega)| \leq 2C(\omega)\left(1+\sum_{n=1}^{N} n2^{\frac{n-1}{2}}|t-s| + \sum_{n=N+1}^{+\infty} n2^{-\frac{n-3}{2}}\right).$$

On the other hand we have that

$$\sum_{n=1}^{N} n2^{\frac{n-1}{2}}|t-s| \leq |t-s|^{\alpha}\sum_{n=1}^{N} n2^{\frac{n-1}{2}}2^{-(1-\alpha)(N-1)} \leq |t-s|^{\alpha}\sum_{n=1}^{N} n2^{-\left(\frac{1}{2}-\alpha\right)(n-1)}.$$

Furthermore we obtain

$$\sum_{n=N+1}^{+\infty} n2^{-\frac{n-3}{2}} = \sum_{n=N+1}^{+\infty} \frac{n2^{-\frac{n-3}{2}}|t-s|^{\alpha}}{|t-s|^{\alpha}} \leq |t-s|^{\alpha}\sum_{n=N+1}^{+\infty} n2^{-\frac{n-3}{2}}2^{\alpha N} \leq 2|t-s|^{\alpha}\sum_{n=N+1}^{+\infty} n2^{-\left(\frac{1}{2}-\alpha\right)\frac{n-1}{2}}$$

This implies that

$$\frac{|B_t(\omega)-B_s(\omega)|}{|t-s|^{\alpha}} \leq 4C(\omega)\left(1+\sum_{n=1}^{N} n2^{-\left(\frac{1}{2}-\alpha\right)\frac{n-1}{2}}\right) < +\infty$$

almost surely. Taking the sup over $0 \leq s < t \leq 1$ the thesis follows. □

3.3 The law of iterated logarithm

Khintchine's version of the law of the iterated logarithm is a more precise statement on the local regularity of a typical Brownian path at a fixed time. It implies in particular that almost every Brownian path is not Hölder continuous with parameter $\alpha \geq \frac{1}{2}$.

Theorem 13. *For $s \geq 0$, the following statements hold almost surely*

$$\limsup_{t\downarrow 0} \frac{|B_{t+s} - B_s|}{\sqrt{2t\log\left(\log\left(\frac{1}{t}\right)\right)}} = 1 \qquad \liminf_{t\downarrow 0} \frac{|B_{t+s} - B_s|}{\sqrt{2t\log\left(\log\left(\frac{1}{t}\right)\right)}} = -1.$$

4 Donsker's Theorem

In this section we show how it is possible to approximate a Brownian motion through a random walk. Let $X_1, \ldots, X_n, \ldots$ be a sequence of random variables independent and identically distributed. We introduce the random walk S_n as the stochastic process defined as follows

$$S_n = \sum_{i=1}^{n} X_i.$$

It is possible to extend the discrete time process S_n to a continuous time process S_t, pathwise continuous, as follows

$$S_t = \begin{cases} S_n & \text{if } t = n \in \mathbb{N} \\ S_{n-1} + (t - n + 1)(S_n - S_{n-1}) & \text{if } n - 1 < t < n \end{cases}.$$

If the random variables X_i are in $L^2(\Omega)$ with mean 0 and variance $\mathbb{E}[X_i^2] = \sigma^2$ we define the following stochastic process

$$\tilde{S}_t^n = \frac{1}{\sqrt{\sigma^2 n}} S_{nt}.$$

Definition 14. *A stochastic process $Y: [0,1] \times \Omega \to \mathbb{R}$ is said to be defined on $C^0([0,1], \mathbb{R})$ if the functions $t \longmapsto Y_t(\omega)$ are continuous for almost every $\omega \in \Omega$.*

Definition 15. *Let $Y_t, Y_t^1, \ldots, Y_t^n, \ldots$ be a sequence of continuous time $t \in [0,1]$ stochastic processes defined on $C^0([0,1], \mathbb{R})$. We say that the sequence of stochastic processes Y_t^n converges to Y in distribution on $C^0([0,1], \mathbb{R})$ if for any bounded and continuous functional $F: C^0([0,1], \mathbb{R}) \to \mathbb{R}$ (where $C^0([0,1], \mathbb{R})$ is equipped with the topology induced by the uniform convergence) we have that*

$$\mathbb{E}[F(Y^n)] \to \mathbb{E}[F(Y)]$$

as $n \to +\infty$.

Remark 16. It is important to note that the convergence in distribution of the finite dimensional marginals $(Y^i_{t_1}, \ldots, Y^i_{t_k})$, of the sequence of processes Y^i_t, to the finite dimensional marginal $(Y_{t_1}, \ldots, Y_{t_k})$ of the process Y, is only a necessary but not sufficient condition for the convergence in distribution on $C^0([0,1],\mathbb{R})$ of Y^i_t to Y.

In the rest of the section we want to prove the following theorem.

Theorem 17. (Donsker's Theorem) *Suppose that $X_i \in L^4(\Omega)$, then we have that $\tilde{S}^n_t$ converges to a Brownian motion B_t on $C^0([0,1],\mathbb{R})$.*

Remark 18. The actual Donsker theorem requires only that $X_i \in L^2(\Omega)$. We assume $X_i \in L^4(\Omega)$ in order to simplify the proof.

4.1 Convergence of finite dimensional distribution

Lemma 19. *Under the hypotheses of Theorem 17, let $0 \leq t_1 \leq \ldots \leq t_k \leq 1$ then $\left(\tilde{S}^n_{t_1}, \tilde{S}^n_{t_2}, \ldots, \tilde{S}^n_{t_k}\right)$ converges in distribution, as $\mathbb{R}^k$ random variables, to $(B_{t_1}, \ldots, B_{t_k})$, where B_t is a Brownian motion.*

Proof. We give the proof only for the case $k=2$, being the general case a simple generalization.

For $k=2$, the thesis of the lemma is equivalent to prove that $(\tilde{S}^n_{t_1}, \tilde{S}^n_{t_2} - \tilde{S}^n_{t_1})$ converges in distribution to a pair of independent random variables with Gaussian distribution and variance t_1 and $t_2 - t_1$, respectively.

First of all we note that $\tilde{S}^n_{t_1} - \frac{1}{\sqrt{\sigma^2 n}} S_{\lfloor nt_1 \rfloor}$ and $\tilde{S}^n_{t_2} - \tilde{S}^n_{t_1} - \frac{1}{\sqrt{\sigma^2 n}}(S_{\lfloor nt_2 \rfloor} - S_{\lceil nt_1 \rceil})$ converges to 0 in L^2. Indeed $\mathbb{E}\left[\left(\tilde{S}^n_{t_1} - \frac{1}{\sqrt{\sigma^2 n}} S_{\lfloor nt_1 \rfloor}\right)^2\right] = \frac{1}{\sqrt{\sigma^2 n}} \mathbb{E}\left[\left(t_1 - \frac{\lfloor nt_1 \rfloor}{n}\right)^2 X^2_{\lceil t_1 n \rceil}\right] \leq \frac{\sigma^2}{\sqrt{\sigma^2 n}} \rightarrow 0$, and a similar relation holds for $\tilde{S}^n_{t_2} - \tilde{S}^n_{t_1} - (S_{\lfloor nt_2 \rfloor} - S_{\lceil nt_1 \rceil})$.

This implies that if $\left(\frac{1}{\sqrt{\sigma^2 n}} S_{\lfloor nt_1 \rfloor}, \frac{1}{\sqrt{\sigma^2 n}}(S_{\lfloor nt_2 \rfloor} - S_{\lceil nt_1 \rceil})\right)$ converges in distribution to a pair of independent random variables with Gaussian distribution and variance t_1 and $t_2 - t_1$, the lemma is proven.

By the Central Limit Theorem, we have that

$$\frac{1}{\sqrt{\sigma^2 n}} S_{\lfloor nt_1 \rfloor} = \sqrt{\frac{\lfloor nt_1 \rfloor}{n}} \cdot \left(\frac{1}{\sqrt{\sigma^2 \lfloor nt_1 \rfloor}} \left(\sum_{k=1}^{\lfloor nt_1 \rfloor} X_i\right)\right) \longrightarrow_d N(0, t_1).$$

In a similar way it is possible to prove that $\frac{1}{\sqrt{\sigma^2 n}}(S_{\lfloor nt_2 \rfloor} - S_{\lceil nt_1 \rceil}) \longrightarrow_d N(0, t_2 - t_1)$. Furthermore since $\frac{1}{\sqrt{\sigma^2 n}} S_{\lfloor nt_1 \rfloor}$ is independent of $\frac{1}{\sqrt{\sigma^2 n}}(S_{\lfloor nt_2 \rfloor} - S_{\lceil nt_1 \rceil})$, the limit of these two random variables is a pair of independent random variables. □

4.2 Convergence in distribution on $C^0([0,1],\mathbb{R})$

Definition 20. *A sequence of stochastic processes Y^n_t defined on $C^0([0,1],\mathbb{R})$ is tight if, for any $\varepsilon > 0$, there exists a compact set $K \subset C^0([0,1],\mathbb{R})$ (with respect to the topology induced by uniform convergence) such that $\mathbb{P}(Y^n \in K) \geq 1 - \varepsilon$ uniformly on $n \in \mathbb{N}$.*

Theorem 21. (Prokhorov's theorem) *Let* $Y, Y_t^1, \ldots, Y_t^n, \ldots$ *be a sequence of stochastic processes defined on* $C^0([0,\ 1],\ \mathbb{R})$*, then the sequence* $Y_t^1,\ \ldots,\ Y_t^n,\ \ldots$ *converges to Y in distribution on* $C^0([0,1],\mathbb{R})$ *if and only if the finite dimensional marginals* $(Y_{t_1}^i, \ldots, Y_{t_k}^i)$*, of the sequence* $\{Y_t^i\}_{i\in\mathbb{N}}$*, converge to the finite dimensional marginals* $(Y_{t_1}, \ldots, Y_{t_k})$ *of the process Y and the sequence* Y^i *is tight.*

Definition 22. *Consider* $K \subset C^0([0,1],\mathbb{R})$*, we say that the functions in K are equibounded and equicontinuous if there exists a* $M > 0$ *such that for any* $f \in K$ $\sup_{t\in[0,1]} |f(t)| \leq M$ *and for any* ε *there exists a* $\delta > 0$ *such that* $\sup_{|t-s|<\delta, t,s\in[0,1]} |f(t) - f(s)| < \varepsilon$.

Theorem 23. (Arzelà-Ascoli Theorem) *A set* $K \subset C^0([0,1],\mathbb{R})$ *is compact (with respect to the topology induced by uniform convergence) if and only if K is closed and the functions defined on K are equibounded and equicontinuous.*

We introduce the following notation, for $f \in C^0([0,1],\mathbb{R})$ and $\delta > 0$ we write

$$w_f(\delta) = \sup_{|t-s|<\delta, t,s\in[0,1]} |f(t) - f(s)|.$$

We also write, for any $t \in [1, 1-\delta]$,

$$\tilde{w}_{f,t}(\delta) = \sup_{s\in[t,t+\delta]} |f(s) - f(t)|.$$

Lemma 24. *Let* $Y_t^1, \ldots, Y_t^n, \ldots$ *be a sequence of stochastic processes defined on* $C^0([0,1],\mathbb{R})$ *such that* $Y_0^n = 0$*. If, for any* $\varepsilon > 0$*, we have*

$$\lim_{\delta\to 0} \left(\limsup_{n\to+\infty} \mathbb{P}(w_{Y^n}(\delta) > \varepsilon) \right) = 0 \tag{7}$$

then $Y_t^1, \ldots, Y_t^n, \ldots$ *is tight.*

Proof. If the limit (7) holds, then for any sequence $\varepsilon_k \to 0$, as $k \to \infty$, and for any $\eta > 0$ there exist two sequences $\delta_k \to 0$ and $n_k \in \mathbb{N}$ such that

$$\mathbb{P}(w_{Y^N}(\delta_k) > \varepsilon_k) \leq 2^{-k}\eta$$

when $N \geq n_k$. Consider the sets $A_k \subset C^0([0,1],\mathbb{R})$ defined as

$$A_k = \{f, f(0) = 0 \,\text{and}\, w_f(\delta_k) \leq \varepsilon_k\}.$$

We have that A_k are closed, and thus the set $A = \cap_{k\in\mathbb{N}} A_k$ is closed. Furthermore the functions in A are equibounded. Indeed if $f \in A$ we have that

$$\sup_{t\in[0,1]} |f(t)| = \sum_{i=0}^{\left\lfloor \frac{1}{\delta_1} \right\rfloor} \sup_{t\in\left[\frac{i}{\delta_1}, \frac{i+1}{\delta_1}\wedge 1\right]} \left| f(t) - f\left(\frac{i}{\delta_1}\right) \right| \leq \left\lceil \frac{i}{\delta_1} \right\rceil \varepsilon_1.$$

By construction A is formed by equicontinuous functions. Thus, by Arzelà-Ascoli Theorem, A is compact. On the other hand

$$\mathbb{P}(Y^n \in A) \geq 1 - \sum_k \mathbb{P}(Y^n \in A_k^c) = 1 - \sum_k \mathbb{P}(w_{Y^N}(\delta_k) > \varepsilon_k) \geq 1 - \eta.$$

Since η is arbitrary the sequence $Y_t^1, \ldots, Y_t^n, \ldots$ is tight. □

Lemma 25. *Let $Y_t^1, \ldots, Y_t^n, \ldots$ be a sequence of stochastic processes defined on $C^0([0,1], \mathbb{R})$ such that $Y_0^n = 0$. If, for any $\varepsilon > 0$, we have*

$$\lim_{\delta \to 0} \left(\limsup_{n \to +\infty} \left(\frac{1}{\delta} \sup_{t \in [0, 1-\delta]} \mathbb{P}(\tilde{w}_{Y^n, t}(\delta) > \varepsilon) \right) \right) = 0 \tag{8}$$

then $Y_t^1, \ldots, Y_t^n, \ldots$ is tight.

Proof. We want to prove that

$$\mathbb{P}(w_{Y^n}(\delta) > 3\varepsilon) \leq \left\lfloor \frac{1}{\delta} \right\rfloor \sup_{t \in [0, 1-\delta]} \mathbb{P}(\tilde{w}_{Y^n, t}(\delta) > \varepsilon).$$

Fix $\delta > 0$ and $\varepsilon > 0$, and consider the set

$$B_t = \{f, f(0) = 0 \text{ and } \tilde{w}_{f,t}(\delta) > \varepsilon\}$$

and consider

$$B = \bigcup_{i < \delta^{-1}} B_{i \cdot \delta}$$

then we have that $\{f, w_f(\delta) > 3\varepsilon\} \subset B$. Indeed suppose that $s \leq t \in [0, 1]$ realizes the sup of for $w_f(\delta)$, then there are two $i_1, i_2 \in \mathbb{N}$ (equal or one next to the other) such that $t \in \left[\frac{i_1}{\delta}, \frac{i_1+1}{\delta}\right]$ and $s \in \left[\frac{i_2}{\delta}, \frac{i_2+1}{\delta}\right]$. Then we have that

$$\left| f(t) - f\left(\frac{i_1}{\delta}\right) \right| + \left| f(s) - f\left(\frac{i_2}{\delta}\right) \right| + \left| f\left(\frac{i_2}{\delta}\right) - f\left(\frac{i_1}{\delta}\right) \right| \geq |f(t) - f(s)| \geq 3\varepsilon$$

which implies that one term among $\left| f(t) - f\left(\frac{i_1}{\delta}\right) \right|$, $\left| f(s) - f\left(\frac{i_2}{\delta}\right) \right|$, and $\left| f\left(\frac{i_2}{\delta}\right) - f\left(\frac{i_1}{\delta}\right) \right|$ is greater than ε, which means that $\{f, w_f(\delta) > 3\varepsilon\} \subset B$.

On the other hand we have

$$\mathbb{P}(Y^n \in B_{i \cdot \delta}) \leq \sup_{t \in [0, 1-\delta]} \mathbb{P}(\tilde{w}_{Y^n, t}(\delta) > \varepsilon).$$

Thus we obtain

$$\mathbb{P}(w_{Y^n}(\delta) > \varepsilon) \leq \sum_{i \leq \delta^{-1}} \mathbb{P}(Y^n \in B_{i \cdot \delta}) \leq \sum_{i \leq \delta^{-1}} \sup_{t \in [0, 1-\delta]} \mathbb{P}(\tilde{w}_{Y^n, t}(\delta) > \varepsilon) \leq \left\lfloor \frac{1}{\delta} \right\rfloor \sup_{t \in [0, 1-\delta]} \mathbb{P}(\tilde{w}_{Y^n, t}(\delta) > \varepsilon)$$

and, using Lemma 24, the lemma is proved. □

4.3 Proof of Donsker's Theorem

Proof of Theorem 17. By Prokhorov's Theorem we have to prove that the finite dimensional marginals converge and that the sequence $\tilde{S}^1, \ldots, \tilde{S}^n$ is tight. The convergence of finite dimensional marginals is proven in Lemma 19. For proving the tightness of the sequence $\tilde{S}^1, \ldots, \tilde{S}^n$ we use Lemma 25.

First of all we note that for any $t \in [0, 1-\delta]$

$$\tilde{w}_{\tilde{S}^n,t}(\delta) \leq \frac{1}{\sqrt{\sigma^2 n}} \left(\sup_{k \in \{\lfloor nt \rfloor, \ldots, \lceil n(t+\delta) \rceil\}} |S_k - S_{\lfloor nt \rfloor}| \right),$$

since the S_t is the straight line interpolation between $\{S_k\}_{k \in \mathbb{N}}$. This means that

$$\mathbb{P}(\tilde{w}_{\tilde{S}^n,t}(\delta) > \epsilon) \leq \mathbb{P}\left(\sup_{k \in \{\lfloor nt \rfloor, \ldots, \lceil n(t+\delta) \rceil\}} |S_k - S_{\lfloor nt \rfloor}| > \sqrt{\sigma^2 n} \cdot \varepsilon \right).$$

Since X_i are i.i.d. we have that

$$\mathbb{P}\left(\sup_{k \in \{\lfloor nt \rfloor, \ldots, \lceil n(t+\delta) \rceil\}} |S_k - S_{\lfloor nt \rfloor}| > \sqrt{\sigma^2 n} \cdot \varepsilon \right) = \mathbb{P}\left(\sup_{k \in \{1, \ldots, \lceil n\delta \rceil\}} |S_k| > \sqrt{\sigma^2 n} \cdot \varepsilon \right).$$

The sequence $S_1, \ldots, S_n, \ldots$ is a martingale with respect to its natural filtration, since it is the sum of i.i.d. random variables with zero mean. Thus, by Doob inequality for martingales, we have

$$\mathbb{P}\left(\sup_{k \in \{1, \ldots, \lceil n\delta \rceil\}} |S_k| > \sqrt{\sigma^2 n} \cdot \varepsilon \right) \leq \frac{\mathbb{E}[S^4_{\lceil n\delta \rceil}]}{\sigma^4 n^2 \epsilon^4}.$$

On the other hand, since X_i are i.i.d. random variables with zero mean we have

$$\mathbb{E}[S_k^4] = \sum_{i,j \leq k} \mathbb{E}[X_i^2 X_j^2] \leq k^2 \mathbb{E}[X_1^4].$$

This implies that

$$\mathbb{P}\left(\sup_{k \in \{1, \ldots, \lceil n\delta \rceil\}} |S_k| > \sqrt{\sigma^2 n} \cdot \varepsilon \right) \leq \frac{\lceil n\delta \rceil^2 \mathbb{E}[X_1^4]}{\sigma^4 n^2 \epsilon^4}.$$

Thus

$$\begin{aligned}
& \lim_{\delta \to 0} \left(\limsup_{n \to +\infty} \left(\frac{1}{\delta} \sup_{t \in [0, 1-\delta]} \mathbb{P}(\tilde{w}_{\tilde{S}^n,t}(\delta) > \varepsilon) \right) \right) \\
\leq\ & \lim_{\delta \to 0} \left(\limsup_{n \to +\infty} \left(\frac{1}{\delta} \mathbb{P}\left(\sup_{k \in \{1, \ldots, \lceil n\delta \rceil\}} |S_k| > \sqrt{\sigma^2 n} \cdot \varepsilon \right) \right) \right) \\
\leq\ & \lim_{\delta \to 0} \left(\limsup_{n \to +\infty} \left(\frac{1}{\delta} \frac{\lceil n\delta \rceil^2 \mathbb{E}[X_1^4]}{\sigma^4 n^2 \epsilon^4} \right) \right) \\
=\ & 0.
\end{aligned}$$

So, by using Lemma 25, the thesis follows. □