Note 3

Martingales

see also A. Bovier's script for SS17, Chapter 2 [pdf].

1 Filtrations and stopping times

As usual we fix a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A (discrete time) stochastic process $(X_n)_{n \geqslant 0}$ is a countable family of random variables indexed by $\mathbb{N}$. Alternatively we can consider it as a measurable map $X: \Omega \times \mathbb{N} \to \mathbb{R}$. For all $\omega \in \Omega$, $X(\omega) \in \mathbb{R}^{\mathbb{N}}$ is a sequence of real numbers.

Definition 1. *A filtration is a family $(\mathscr{F}_n)_{n \geqslant 0}$ of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ for all $n \geqslant 0$. We will always let $\mathscr{F}_{-1} = \{\emptyset, \Omega\}$ and $\mathscr{F}_\infty = \sigma(\mathscr{F}_n, n \geqslant 0)$ ($\mathscr{F}_\infty$ is the smallest σ-algebra which contains all the $\mathscr{F}_n$ for $n \geqslant 0$). Let $(X_n)_{n \geqslant 0}$ a stochastic process, then its natural filtration $(\mathscr{F}_n^X)_{n \geqslant 0}$ is the filtration defined by $\mathscr{F}_n^X = \sigma(X_0, \ldots, X_n)$.*

Definition 2. *Let $(X_n)_{n \geqslant 0}$ a stochastic process and $(\mathscr{F}_n)_{n \geqslant 0}$ a filtration. We say that $(X_n)_{n \geqslant 0}$ is adapted (to the filtration $(\mathscr{F}_n)_{n \geqslant 0}$) iff $X_n \hat{\in} \mathscr{F}_n$ for all $n \geqslant 0$. We say that $(X_n)_{n \geqslant 0}$ is previsible (with respect to the filtration $(\mathscr{F}_n)_{n \geqslant 0}$) iff $X_n \hat{\in} \mathscr{F}_{n-1}$ for all $n \geqslant 0$. The natural filtration of X is the smallest filtration which makes X adapted.*

A filtration $(\mathscr{F}_n)_{n \geqslant 0}$ represents information gathered along the flow of time. An adapted process $(X_n)_{n \geqslant 0}$ is a process which we discover progressively: at time $n \geqslant 0$ we dispose only of the information in $\mathscr{F}_n$ and therefore only the values of X_k for $k \leqslant n$ (the past of n) are known and not the values of X_k for $k > n$ (the future of n).

Example 3. (Random walk) Let $(X_n)_{n \geqslant 1}$ a sequence of i.i.d. random variables and $(\mathscr{F}_n)_{n \geqslant 0}$ its natural filtration (i.e. $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$, $\mathscr{F}_0 = \{\emptyset, \Omega\}$). We let $S_n = S_0 + X_1 + \cdots + X_n$ with $S_0 \hat{\in} \mathscr{F}_0$ (a constant). Then $(S_n)_{n \geqslant 0}$ is a process adapted to $(\mathscr{F}_n)_{n \geqslant 0}$.

We have also $\mathscr{F}_n = \sigma(S_1, \ldots, S_n)$.

Let us now play heads and tails betting €1 each time and let X_n represent the gain on the n-th game: $\mathbb{P}(X_n = \pm 1) = 1/2$. The process $S_n = S_0 + X_1 + \cdots + X_n$ represents then the total gain after the firsts n games. We allow $S_n < 0$: in this case we say that $(S_n)_-$ is the money we borrowed to continue to play.

Obviously in a fair game the average gain is zero: $\mathbb{E}[S_n] = S_0 + n\,\mathbb{E}[X_1] = S_0$.

Let us now allow to quit the game at a time which depends on the outcomes of the game itself. Otherwise said, quit the game at a random time $T: \Omega \to \mathbb{N}^* = \mathbb{N} \cup \{+\infty\}$ ($T = +\infty$ means that we are not actually quitting and we continue to play forever). Is clear that we should not allow any r.v. T as a stopping strategy. Let us see some examples:

1. I quit as soon as I lose the first time: $T_1 = \inf\{n \geqslant 1: X_n = -1\}$;

2. I quit as soon as I win at least €100: $T_2 = \inf\{n \geqslant 1 : S_n \geqslant S_0 + 100\}$;

3. I quit just before loosing the first time: $T_3 = \inf\{n \geqslant 0 : X_{n+1} = -1\}$.

The first two strategies are acceptables, while the third not: usually it would require a knowledge of the future. The first two strategies are examples of *stopping times*, according the the following definition.

Definition 4. *A r.v.* $T\colon \Omega \to \mathbb{N}_* = \mathbb{N} \cup \{+\infty\}$ *is a stopping time (for the filtration* $(\mathscr{F}_n)_n$*) iff* $\{T \leqslant n\} \in \mathscr{F}_n$ *for all* $0 \leqslant n \leqslant +\infty$*. Equivalently, T is a stopping time iff* $\{T = n\} \in \mathscr{F}_n$ *for all* $0 \leqslant n \leqslant +\infty$.

Remark 5.

Example 6. Let $(X_n)_{n\geqslant 1}$ an adapted process and A a Borel set of $\mathbb{R}$, then

$$T_A = \inf\{n \geqslant 1 : X_n \in A\},$$

(with $\inf(\emptyset) = +\infty$) is a stopping time: for all $0 \leqslant n \leqslant +\infty$ we have $\{T \leqslant n\} = \cup_{0<k\leqslant n}\{X_k \in A\} \in \mathscr{F}_n$.

Exercise 1. Show that T_2 is a stopping time and that T_3 is not.

If $(X_n)_{n\geqslant 0}$ is a stochastic process we denote $X_T\colon \Omega \to \mathbb{R}$ the random variable given by

$$X_T(\omega) = X_{T(\omega)}(\omega), \qquad \omega \in \Omega.$$

Exercise 2. Show that if T is a stopping time and $(X_n)_{n\geqslant 0}$ an adapted process, then the process $X_n^T(\omega) = X_{n\wedge T(\omega)}(\omega)$ is also adapted. It is called the process stopped in T.

We will need also the notion of all the information available at a given stopping time T.

Definition 7. *Let*

$$\mathscr{F}_T := \{A \in \mathscr{F} : A \cap \{T \leqslant n\} \in \mathscr{F}_n \text{ for all } n \in \mathbb{N}_*\}.$$

Then $\mathscr{F}_T$ *is a* σ*-algebra.*

Proposition 8. *Let* S, T *two stopping times.*

a) If $S \leqslant T$ *then* $\mathscr{F}_S \subseteq \mathscr{F}_T$*;*

b) $\mathscr{F}_{S\wedge T} = \mathscr{F}_T \cap \mathscr{F}_S$*,* $\mathscr{F}_{T\vee S} = \sigma(\mathscr{F}_T, \mathscr{F}_S)$*;*

c) If $(X_n)_n$ *is an* $(\mathscr{F}_n)_n$*-adapted process, then* $X_T \mathrel{\hat{\in}} \mathscr{F}_T$*.*

d) $Z \mathrel{\hat{\in}} \mathscr{F}_T$ *if and only if the process* $(Z_n = Z\mathbb{1}_{\{T=n\}})_{n\in N_*}$ *is adapted, moreover* $Z_T = Z$*.*

Proof. Exercise. □

If we use a stopping time T in the heads and tail game above we obtain a final gain S_T. The the following result applies.

Theorem 9. *(Wald's identity) Let $(X_n)_{n\geqslant 1}$ an i.i.d. sequence of r.v. such that $\mathbb{E}[|X_1|]<+\infty$ and T a stopping time for the filtration generated by X. If $\mathbb{E}[T]<+\infty$ then*

$$\mathbb{E}[S_T]=\mathbb{E}[T]\,\mathbb{E}[X_1],$$

where $S_n=X_1+\cdots+X_n$.

Proof. We note that

$$S_T(\omega)=\sum_{n\geqslant 1} X_n(\omega)\mathbb{I}_{n\leqslant T(\omega)}, \qquad T(\omega)=\sum_{n=1}^{T(\omega)} 1=\sum_{n\geqslant 1}\mathbb{I}_{n\leqslant T(\omega)},$$

the sums being finite a.s. since $\mathbb{E}[T]<+\infty$ and therefore $\mathbb{P}(T=+\infty)=0$. We have

$$|S_T|\leqslant \sum_{n\geqslant 1}|X_n|\mathbb{I}_{n\leqslant T}.$$

Fubini's theorem gives

$$\mathbb{E}[\sum_{n\geqslant 1}|X_n|\mathbb{I}_{n\leqslant T}]=\sum_{n\geqslant 1}\mathbb{E}[|X_n|\mathbb{I}_{n\leqslant T}].$$

Since T is a stopping time, we have $\{T\geqslant n\}=\{T<n\}^c=\{T\leqslant n-1\}^c\in\mathscr{F}_{n-1}$ and by the properties of conditional expectation

$$\sum_{n\geqslant 1}\mathbb{E}[|X_n|\mathbb{I}_{n\leqslant T}]=\sum_{n\geqslant 1}\mathbb{E}[\mathbb{E}[|X_n|\underbrace{\mathbb{I}_{n\leqslant T}}_{\in\mathscr{F}_{n-1}}|\mathscr{F}_{n-1}]]=\sum_{n\geqslant 1}\mathbb{E}[\mathbb{E}[\underbrace{|X_n|}_{\perp\mathscr{F}_{n-1}}|\mathscr{F}_{n-1}]\mathbb{I}_{n\leqslant T}]$$

$$=\sum_{n\geqslant 1}\mathbb{E}[|X_n|]\mathbb{E}[\mathbb{I}_{n\leqslant T}]=\mathbb{E}[|X_1|]\mathbb{E}[\sum_{n\geqslant 1}\mathbb{I}_{n\leqslant T}]=\mathbb{E}[|X_1|]\mathbb{E}[T]$$

which shows that the map $(\omega,n)\mapsto|X_n(\omega)|\mathbb{I}_{1\leqslant n\leqslant T(\omega)}$ is integrable w.r.t. the product measure $\mathbb{P}\times\mathbb{Q}$ on $\Omega\times\mathbb{N}$ where $\mathbb{Q}$ is the counting measure of $\mathbb{N}$. Then we can use Fubini-Tonelli on the function $(\omega,n)\mapsto X_n(\omega)\mathbb{I}_{1\leqslant n\leqslant T(\omega)}$ and by the same reasoning we obtain

$$\mathbb{E}[S_T]=\mathbb{E}[\sum_{n\geqslant 1}X_n\mathbb{I}_{n\leqslant T}]\overset{\text{Fubini}}{=}\sum_{n\geqslant 1}\mathbb{E}[X_n\mathbb{I}_{n\leqslant T}]\overset{\text{independence}}{=}\sum_{n\geqslant 1}\mathbb{E}[X_n]\mathbb{E}[\mathbb{I}_{n\leqslant T}]=\mathbb{E}[X_1]\mathbb{E}[T].$$

□

Wald's identity applied to our game of heads and tails give us that as long as our strategy is given by a stopping time then the average gain is always 0.

Remark 10. The integrability condition on T in Wald's identity is fundamental. Consider the stopping time $T=T_2=\inf\{n\geqslant 1: S_n\geqslant S_0+100\}$. Then by definition $T<+\infty\Rightarrow S_T=S_0+100$ so if we could apply the identity we would obtain $S_0=\mathbb{E}[S_T]=S_0+100$! This shows that we need to have necessarily $\mathbb{E}[T]=+\infty$. (Since if $\mathbb{E}[T]<+\infty$ then $T<+\infty$ a.s. and $S_T=S_0+100$ a.s.).

Remark 11. In general, if the process $(X_n)_{n\geqslant 0}$ is adapted and integrable (i.e. $X_n\in L^1$ for all $n\geqslant 0$) and T is a bounded stopping time (i.e. there exists an integer $N<+\infty$ such that $\mathbb{P}(T\leqslant N)=1$) then $X_T\in L^1$ since

$$|X_T|=|\sum_{n=1}^N X_n \mathbb{I}_{T=n}|\leqslant \sum_{n=1}^N |X_n|\,\mathbb{I}_{T=n}\in L^1$$

being finite sum of r.v. in L^1.

Given these observations is interesting to study the class $\mathscr{M}$ of adapted and integrable stochastic processes $(X_n)_{n\geqslant 0}$ such that

$$\mathbb{E}[X_T]=X_0 \qquad \text{for all bounded stopping time } T. \tag{1}$$

The class $\mathscr{M}$ intuitively represent the class of processes of gains in fair games that do not admit a stopping strategy which is profitable in the average. By Wald's identity, all the partial sums of i.i.d. random variables with zero mean belongs to this class. It is then interesting to investigate the general properties of proecsses belonging to $\mathscr{M}$.

A property which plays a fundamental role in the caracterisation and study of such a class is the following.

Lemma 12. *An adapted and integrable process $(X_n)_{n\geqslant 0}$ satisfies (1) iff for all $n\geqslant 0$,*

$$\mathbb{E}[X_{n+1}|\mathscr{F}_n]=X_n \tag{2}$$

Proof. Let us show that (1) ⇒ (2). For all $n\geqslant 0$ and $A\in\mathscr{F}_n$ consider the stopping time (check that it is such indeed)

$$T_{n,A}(\omega)=\begin{cases} n+1 & \text{si } \omega\in A\\ n & \text{sinon}\end{cases}$$

then the condition $\mathbb{E}[X_{T_{n,A}}]-X_0=0$ gives

$$0=\mathbb{E}[X_{n+1}\mathbb{I}_A+X_n(1-\mathbb{I}_A)]-X_0=\mathbb{E}[(X_{n+1}-X_n)\mathbb{I}_A]+\mathbb{E}[X_n]-X_0=\mathbb{E}[(X_{n+1}-X_n)\mathbb{I}_A]$$

(since $\mathbb{E}[X_n]=X_0$ by definition of the class $\mathscr{M}$). This last equality holds for all $A\in\mathscr{F}_n$ which gives that $\mathbb{E}[X_{n+1}-X_n|\mathscr{F}_n]=0$ a.s.

Let us show now that (2) ⇒ (1). Let T be a bounded stopping time and N an integer such that $T\leqslant N$. Remark that condition (2) implies for all $k>n$

$$\mathbb{E}[X_k|\mathscr{F}_n]=\mathbb{E}[\mathbb{E}[X_k|\mathscr{F}_{k-1}]|\mathscr{F}_n]=\mathbb{E}[X_{k-1}|\mathscr{F}_n]=\cdots=\mathbb{E}[X_n|\mathscr{F}_n]=X_n,$$

since $\mathscr{F}_k\supseteq\mathscr{F}_n$ if $k\geqslant n$. Therefore $\mathbb{E}[X_N|\mathscr{F}_n]=X_n$ for all $n\leqslant N$ and also $\mathbb{E}[X_N]=X_0$ by taking the expectation of both sides with $n=0$. Thanks to the integrability hypothesis on X_n and boundedness of T we have

$$\mathbb{E}[X_T]=\sum_{n=1}^N \mathbb{E}[X_n\mathbb{I}_{n=T}]\overset{\text{eq.(2)}}{=}\sum_{n=1}^N \mathbb{E}[\mathbb{E}[X_N|\mathscr{F}_n]\underbrace{\mathbb{I}_{n=T}}_{\in\mathscr{F}_n}]=\sum_{n=1}^N \mathbb{E}[\mathbb{E}[X_N\mathbb{I}_{n=T}|\mathscr{F}_n]]$$

$$=\sum_{n=1}^N \mathbb{E}[X_N\mathbb{I}_{n=T}]=\mathbb{E}[X_N]=X_0.$$

which gives property (1). □

Eq. (2) can be interpreted in terms of game strategies by saying that in a fair game where $X_{n+1}-X_n$ represents the gain in the $(n+1)$ play, then this gain has zero mean conditionally to the past history of the game. i.e. $\mathbb{E}[X_{n+1}-X_n|\mathscr{F}_n]=0$.

Remark 13. The previous proof shows also that if the condition $\mathbb{E}[X_{n+1}-X_n|\mathscr{F}_n]=0$ is not always verified then we can construct a stopping time T such that $\mathbb{E}[X_T]\neq X_0$.

Indeed assume that there exists $n>0$ for which the event $A=\{\mathbb{E}[X_{n+1}-X_n|\mathscr{F}_n]>0\}$ (a similar reasoning works for the case <0) has $\mathbb{P}(A)>0$. Note that $A\in\mathscr{F}_n$ (check) which means that at time n we already know if we happen to be in the event A or not. The stopping strategy is then the following: if we are in A then we will stop at time $n+1$, otherwise we immediately stop (at time n). The underlying idea is that if we are in A then we know that $\mathbb{E}[X_{n+1}|\mathscr{F}_n]>X_n$ and therefore if we continue to play we will gain something positive on average. We let then

$$T=(n+1)\,\mathbb{I}_A+n\mathbb{I}_{A^c},$$

which is a stopping time. With this stopping time we obtain

$$\mathbb{E}[X_T]=\mathbb{E}[X_{n+1}\mathbb{I}_A+X_n\mathbb{I}_{A^c}]=\mathbb{E}[\mathbb{E}[X_{n+1}|\mathscr{F}_n]\mathbb{I}_A+X_n\mathbb{I}_{A^c}]>\mathbb{E}[X_n\mathbb{I}_A+X_n\mathbb{I}_{A^c}]=\mathbb{E}[X_n]=X_0.$$

The inequality is strict since the r.v. $Q=\mathbb{E}[X_{n+1}|\mathscr{F}_n]\mathbb{I}_A-X_n\mathbb{I}_A\geqslant 0$ and $Q>0$ with positive probabilty $\mathbb{P}(Q>0)=\mathbb{P}(A)>0$. This implies that $\mathbb{E}[Q]>0$ and therefore that $\mathbb{E}[\mathbb{E}[X_{n+1}|\mathscr{F}_n]\mathbb{I}_A]>\mathbb{E}[X_n\mathbb{I}_A]$.

We will call *martingales* the processes satisfying (2) and in the following we will study their general properties.

2 Martingales

Definition 14. *A real, adapted and integrable process* $(X_n)_{n\geqslant 0}$ *(i.e.* $\mathbb{E}[|X_n|]<+\infty$ *for all* $n\geqslant 0$*) is*

i. a martingale iff $\mathbb{E}[X_{n+1}|\mathscr{F}_n]=X_n$ *for all* $n\geqslant 0$*;*

ii. a super-martingale iff $\mathbb{E}[X_{n+1}|\mathscr{F}_n]\leqslant X_n$ *for all* $n\geqslant 0$*;*

iii. a sub-martingale iff $\mathbb{E}[X_{n+1}|\mathscr{F}_n]\geqslant X_n$ *for all* $n\geqslant 0$*.*

Interpreting $(X_n)_{n\geqslant 0}$ as the gain in a game and the filtration $(\mathscr{F}_n)_{n\geqslant 0}$ as the information available at any given time, then a martingale correspons to a fair game, a super-martingale to a unfavorable game and a sub-martingale to a favorable game.

Remark 15. If X is a martingale, then by recurrence we have that $\mathbb{E}[X_m|\mathscr{F}_n]=X_n$ for all $m\geqslant n\geqslant 0$. A similar property is true for super/sub-martingales. This alternative caracterisation holds also in continuous time. If we denote $\Delta X_n=X_n-X_{n-1}$ then the (sub-/super-)martingale property is equivalent to

$$\mathbb{E}[\Delta X_{n+1}|\mathscr{F}_n]=0\,(\text{or}\geqslant,\text{ or}\leqslant)\text{ for all }n\geqslant 0.$$

Example 16. Let Z be a real and integrable random variable. Then $(X_n = \mathbb{E}[Z|\mathscr{F}_n])_n$ is a martingale. If $(A_n)_{n\geqslant 0}$ is a real, adapted and increasing (resp. decreasing) process, then it is also a sub-martingale (resp. super-martingale).

Example 17. Let $(X_n)_{n\geqslant 1}$ a sequence of r.v. such that $\mathbb{E}[X_n] = 0$ for all $n \geqslant 1$. Let $Y_0 = 0$ and $Y_n = X_1 + \cdots + X_n$ for $n \geqslant 1$. Then $(Y_n)_{n\geqslant 0}$ is a martingale wrt. $(\mathscr{F}_n^X)_{n\geqslant 0}$ and therefore also wrt. $(\mathscr{F}_n^Y)_{n\geqslant 0}$.

Proposition 18. *(Doob's decomposition) Let $(X_n)_{n\geqslant 0}$ an adapted and integrable sequence, then there exists a unique martingale $(M_n)_{n\geqslant 0}$ and a unique integrable and previsible process $(I_n)_{n\geqslant 0}$ such that $I_0 = 0$ and*

$$X_n = X_0 + M_n + I_n, \qquad n \geqslant 0.$$

Moreover

a) $I_n = 0$ for all $n \geqslant 0$ iff $(X_n)_{n\geqslant 0}$ is a martingale;

b) $(I_n)_{n\geqslant 0}$ is increasing iff $(X_n)_{n\geqslant 0}$ is a sub-martingale;

c) $(I_n)_{n\geqslant 0}$ is decreasing iff $(X_n)_{n\geqslant 0}$ is a super--martingale.

Proof. First, lets show uniqueness. If $\tilde{M}, \tilde{I}$ give another possible decomposition of X, then we must have $\tilde{M}_n + \tilde{I}_n = M_n + I_n = X_n - X_0$ and therefore, letting $N_n = \tilde{M}_n - M_n = I_n - \tilde{I}_n$ we have that $(N_n)_n$ is both a martingale and an integrable previsible process, which for all $n \geqslant 0$ implies

$$N_n = \mathbb{E}[N_{n+1}|\mathscr{F}_n] = N_{n+1},$$

since $N_{n+1} \hat{\in} \mathscr{F}_n$. Therefore $(N_n)_n$ is constant in n and $N_n = N_0 = 0$ since $I_0 = \tilde{I}_0 = 0$. We conclude that $I_n = \tilde{I}_n$ and $M_n = \tilde{M}_n$. For the existence part we note that $\Delta M_n = \Delta X_n - \Delta I_n$ and by taking the conditional expectation we get

$$0 = \mathbb{E}[\Delta M_{n+1}|\mathscr{F}_n] = \mathbb{E}[\Delta X_{n+1}|\mathscr{F}_n] - \mathbb{E}[\Delta I_{n+1}|\mathscr{F}_n] = \mathbb{E}[\Delta X_{n+1}|\mathscr{F}_n] - \Delta I_{n+1}$$

since by prevesibility $\Delta I_{n+1} \hat{\in} \mathscr{F}_n$. We can set

$$I_n = \sum_{i=0}^{n-1} \mathbb{E}[\Delta X_{i+1}|\mathscr{F}_i], \qquad I_0 = 0$$

which indeed gives us an integrable and previsible process. Is also clear that if we let $M_n = X_n - X_0 - I_n$ then $(M_n)_{n\geqslant 0}$ is a martingale.

The definition of $(I_n)_n$ gives directly that if $(X_n)_{n\geqslant 0}$ is a martingale then $I_n = 0$ for all $n \geqslant 0$, and the reverse implication is also clear. If $(X_n)_{n\geqslant 0}$ is a super/sub-martingale then for all n: $\mathbb{E}[\Delta X_{n+1}|\mathscr{F}_n] \geqslant X_n$ (or $\leqslant$) and therefore the process $(I_n)_n$ is decreasing/increasing. □

Proposition 19. *Let $(X_n)_{n\geqslant 0}$ be a martingale (resp. sub-martingale) and $\Phi: \mathbb{R} \to \mathbb{R}$ a convex function (resp. convext and increasing) such that $(\Phi(X_n))_n$ is an integrable process. Then $(\Phi(X_n))_n$ is a sub-martingale.*

Proof. By conditional Jensen's inequality

$$\mathbb{E}[\Phi(X_{n+1})|\mathscr{F}_n] \geqslant \Phi(\mathbb{E}[X_{n+1}|\mathscr{F}_n]) = \Phi(X_n)$$

where the last equality follows from the martingale property. If X is a sub-martingale

$$\mathbb{E}[\Phi(X_{n+1})|\mathscr{F}_n] \geqslant \Phi(\mathbb{E}[X_{n+1}|\mathscr{F}_n]) \geqslant \Phi(X_n),$$

since Φ is increasing. □

Proposition 20. *Let $(X_n)_{n\geqslant 0}$ be a square-integrable martingale (i.e. $\mathbb{E}[X_n^2] < +\infty$ for all $n \geqslant 0$). then the sub-martingale $(X_n^2)_{n\geqslant 0}$ has the decomposition*

$$X_n^2 = X_0^2 + N_n + [X]_n$$

where

$$N_n = 2\sum_{i=1}^{n} X_{i-1}\Delta X_i, \qquad [X]_n = \sum_{i=1}^{n} (\Delta X_i)^2.$$

The process $(N_n)_{n\geqslant 0}$ is martingale and $([X]_n)_{n\geqslant 0}$ is an increasing process called the quadratic variation of X.

Proof. (exercise) □

Remark 21. Note that Doob's decomposition of $(X_n^2)_{n\geqslant 0}$ is $X_n^2 = X_0^2 + M_n + \langle X\rangle_n$ where $(\langle X\rangle_n)_{n\geqslant 0}$ is an increasing and previsible process. Then a computation gives

$$\Delta\langle X\rangle_n = \mathbb{E}[(\Delta X_n)^2|\mathscr{F}_{n-1}] = \mathbb{E}[\Delta[X]_n|\mathscr{F}_{n-1}].$$

3 Martingale transforms and optional stopping

Definition 22. *Let $(X_n)_{n\geqslant 0}$ an adapted process and $(C_n)_{n\geqslant 1}$ a previsible one. We define the new adapted process $((C\bullet X)_n)_{n\geqslant 0}$ by $(C\bullet X)_0 = 0$ and $\Delta(C\bullet X)_n = C_n\Delta X_n$ for all $n \geqslant 1$. Then*

$$(C\bullet X)_n = \sum_{i=1}^{n} C_i(X_i - X_{i-1}).$$

Lemma 23. *Let $(C_n)_{n\geqslant 1}$ a bounded previsible process (i.e. $|C_n| \leqslant K < \infty$ for all $n \geqslant 1$ with $K \in \mathbb{R}$).*

i. If $(X_n)_{n\geqslant 0}$ is a martingale then $((C\bullet X)_n)_{n\geqslant 0}$ is also a martingale.

ii. If $(X_n)_{n\geqslant 0}$ is a sub/super-martingale and $C_n \geqslant 0$ for all $n \geqslant 1$ then $((C\bullet X)_n)_{n\geqslant 0}$ is also a sub/super-martingale.

These properties are also valid without the boundedness condition provided $C_n, X_n \in L^2$ for all $n \geqslant 1$.

Proof. The integrability and the adaptedness of $((C\bullet X)_n)_{n\geqslant 0}$ are left as exercise. We have for all $n\geqslant 1$,

$$\mathbb{E}[\Delta(C\bullet X)_n|\mathscr{F}_{n-1}] = \mathbb{E}[C_n\Delta X_n|\mathscr{F}_{n-1}] = C_n\mathbb{E}[\Delta X_n|\mathscr{F}_{n-1}]$$

since $(C_n)_{n\geqslant 1}$ is previsible, then we can conclude. □

Let T be a stopping time and let

$$C_n = 1_{n\leqslant T}$$

then the process $(C_n)_{n\geqslant 1}$ is previsible and $X_0 + (C\cdot X)_n = X_n^T = X_{T\wedge n}$ and we can conclude that:

Theorem 24. *If T is a stopping time and $(X_n)_{n\geqslant 0}$ is a (super-)martingale, then $(X_n^T)_{n\geqslant 0}$ is a (super-)martingale and*

$$\mathbb{E}[X_{n\wedge T}] \leqslant \mathbb{E}[X_0]$$

in the super-martingale case (with equality for a martingale).

Remark 25. No boundedness of T is required here.

Remark 26. Let $(X_n)_{n\geqslant 0}$ the simple random walk on $\mathbb{Z}$ with $X_0 = 0$. Then $(X_n)_{n\geqslant 0}$ is a martingale and for all stopping time T we have $\mathbb{E}[X_{n\wedge T}] = \mathbb{E}[X_0] = 0$. However in general

$$\mathbb{E}[X_T] \neq 0.$$

Indeed, if $T = \inf\{n > 0 : X_n = 1\}$ one can show (later in the course) that $\mathbb{P}(T < +\infty) = 1$ and $X_T = 1$, which imply $\mathbb{E}[X_T] = 1$. Therefore we conclude that the sequence $(X_{T\wedge n})_n$ do not converge to X_T in L^1, but only almost surely.

Now an important generalisation of Wald's identity for (super-)martingales.

Theorem 27. *(Optional stopping theorem) Let T be a stopping time and $(X_n)_{n\geqslant 0}$ a (super-)martingale. Then X_T is integrable and $\mathbb{E}[X_T] \leqslant \mathbb{E}[X_0]$ in the following situations:*

i. T is bounded;

ii. $(X_n)_n$ is uniformly bounded and $T < +\infty$ a.s.;

iii. $\mathbb{E}[T] < +\infty$ and there exists $K > 0$ such that for all $n\geqslant 1$,

$$|X_n - X_{n-1}| \leqslant K.$$

iv. $X_n \geqslant 0$ for all $n\geqslant 0$ and $T < +\infty$ a.s.

Proof. We know that $\mathbb{E}[X_{n\wedge T} - X_0] \leqslant 0$ for all $n\geqslant 1$. (i) If $T\leqslant N$ is enough to take $n = N$. (ii) We can use dominated convergence to show that

$$0 \geqslant \lim_n \mathbb{E}[X_{n\wedge T} - X_0] = \mathbb{E}[\lim_n (X_{n\wedge T} - X_0)] = \mathbb{E}[X_T - X_0].$$

(iii) We have that

$$|X_{n\wedge T}-X_0|\leqslant\sum_{k=1}^{T\wedge n}|\Delta X_k|\leqslant KT,$$

since $|\Delta X_k|\leqslant K$ for all $k\geqslant 0$. Since $\mathbb{E}[T]<+\infty$ we deduce by dominated convergence that $\mathbb{E}[X_T]\leqslant\mathbb{E}[X_0]$. (iv) The sequence $(X_{n\wedge T})_{n\geqslant 0}$ is positive and converges a.s. to X_T. By Fatou's lemma we have that

$$\mathbb{E}[X_0]\geqslant\liminf_n\mathbb{E}[X_{n\wedge T}]\geqslant\mathbb{E}[\liminf_n X_{n\wedge T}]=\mathbb{E}[X_T]. \qquad \square$$

Lemma 28. *Let $(X_n)_{n\geqslant 1}$ be a martingale (resp. sub-, super-) and $T\geqslant S$ two bounded stopping times, then*

$$\mathbb{E}[X_T|\mathscr{F}_S]=X_S. \quad \textit{(resp. } \geqslant, \leqslant\textit{).} \tag{3}$$

Proof. Assume X is a martingale. By the boundedness of the stopping times there exists $N\in\mathbb{N}$ such that $S\leqslant T\leqslant N$. Wew begin by showing that $\mathbb{E}[X_N|\mathscr{F}_T]=X_T$. By the definition of conditional expectation we need to check that $\mathbb{E}[X_N\mathbb{1}_B]=\mathbb{E}[X_T\mathbb{1}_B]$ for all $B\in\mathscr{F}_T$. We have that $B\cap\{T=n\}\in\mathscr{F}_n$ and that

$$\begin{aligned}\mathbb{E}[X_N\mathbb{1}_B]&=\sum_{n=1}^{N}\mathbb{E}[X_N\mathbb{1}_{B\cap\{T=n\}}]=\sum_{n=1}^{N}\mathbb{E}[\mathbb{E}[X_N|\mathscr{F}_n]\mathbb{1}_{B\cap\{T=n\}}]\\&=\sum_{n=1}^{N}\mathbb{E}[X_n\mathbb{1}_{B\cap\{T=n\}}]=\sum_{n=1}^{N}\mathbb{E}[X_T\mathbb{1}_{B\cap\{T=n\}}]=\mathbb{E}[X_T\mathbb{1}_B].\end{aligned}$$

Therefore $\mathbb{E}[X_T|\mathscr{F}_S]=\mathbb{E}[\mathbb{E}[X_N|\mathscr{F}_T]|\mathscr{F}_S]=\mathbb{E}[X_N|\mathscr{F}_S]=X_S$ car $\mathscr{F}_S\subseteq\mathscr{F}_T$.

If $(X_n)_{n\geqslant 1}$ a sub-martingale, then by Doob's decomposition there exists a martingale $(M_n)_{n\geqslant 1}$ and a previsible non-decreasing process $(A_n)_{n\geqslant 1}$ (i.e. $A_n\in\mathscr{F}_{n-1}$ and $A_{n+1}\geqslant A_n$) such that $X_n=M_n+A_n$. We have

$$\mathbb{E}[X_T|\mathscr{F}_S]=\mathbb{E}[M_T|\mathscr{F}_S]+\mathbb{E}[A_T|\mathscr{F}_S]=M_S+\mathbb{E}[A_T|\mathscr{F}_S]\geqslant M_S+\mathbb{E}[A_S|\mathscr{F}_S]=M_S+A_S=X_S$$

since $A_T\geqslant A_S$ due to the fact that $(A_n)_{n\geqslant 1}$ is non-decreasing and that $T\geqslant S$. For supermartingales one reason analogously. □

Remark 29. If F is an integrable random variable then $(F_n=\mathbb{E}[F|\mathscr{F}_n])_n$ is a martingale and the previous lemma implies $\mathbb{E}[F|\mathscr{F}_S]=F_S$ for all bounded stopping times S. Therefore we can compute the conditional expectation wrt. a σ-algebra $\mathscr{F}_S$ by taking the value at time $n=S$ of the conditional expectation computed wrt. the filtration $(\mathscr{F}_n)_n$.