Note 4

Asymptotic behaviour of martingales

see also A. Bovier's script for SS17, Chapter 2 [pdf].

1 Convergence of martingales

Recall that a super-martingale $(X_n)_{n\geqslant 1}$ is the gain in an unfavorable game where in average we expect to lose at each hand: $\mathbb{E}[\Delta X_n|\mathscr{F}_{n-1}]\leqslant 0$.

Let $a<b$ two reals and consider the following playing strategy: we wait the first time S_1 where $X_{S_1}<a$. Then we start to play until the first time $T_1>S_1$ where $X_{T_1}>b$. At this point we won $X_{T_1}-X_{S_1}>b-a$ and subsequently we avoid to play until the time $S_2>T_1$ where X_{S_2} become again $<a$. From this moment on we restart to play our strategy. If we fix a time horizon $n<\infty$ and we denote by $U_n(a,b)$ how many times $(X_k)_{1\leqslant k\leqslant n}$ goes from $(-\infty,a)$ to $(b,+\infty)$ and by W_n our gain using the above strategy, then we surely will have $W_0=0$ and

$$W_n\geqslant(b-a)U_n(a,b)-(X_n-a)_-. \tag{1}$$

The term $(X_n-a)_-$ corresponds to our eventual losses in the last upcrossing before attaining another time the b threshold.

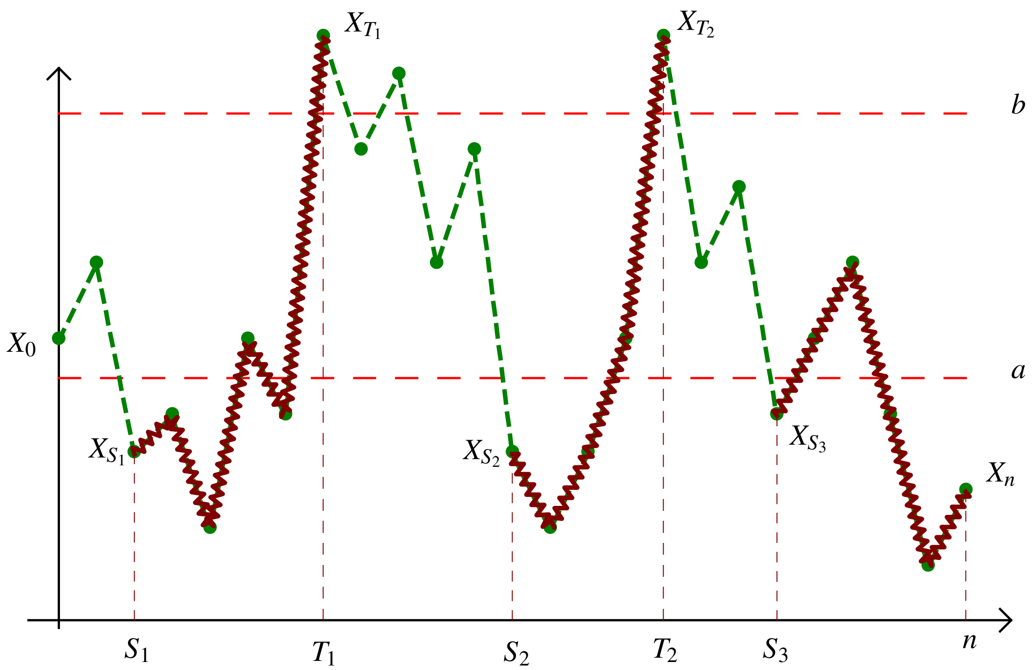

Figure 1. Upcrossings. In this example two upcrossings have been completed, while the third is still going on at time n and in an unfavorable situation, since $X_n\leqslant X_{S_3}\leqslant a$. The gain is therefore $W_n\geqslant 2(b-a)+(X_n-a)$.

Is also easy to see that $(W_n)_{n\geqslant 1}$ is a sur-martingale since we can write

$$W_n = W_0 + \sum_{k=1}^{n} H_n \, \Delta X_n$$

with $H_n = \sum_{i=1}^{\infty} \mathbb{I}_{S_i \leqslant n \leqslant T_i - 1} = \mathbb{I}_{\{n \in \cup_{i\geqslant 1}[S_i, T_i - 1]\}}$. Equivalently we could define $(H_n)_{n\geqslant 1}$ by the recurrence relation:

$$H_{n+1} = \mathbb{I}_{H_n = 0, X_n < a} + \mathbb{I}_{H_n = 1, X_n > b}\,.$$

Then $U_n(a,b) = \sum_{i=2}^{n} \mathbb{I}_{H_n = 0, H_{n-1} = 1}$. We leave to the reader the exercise to show that $(H_n)_{n\geqslant 1}$ is a previsible process.

We can show that eq. (1) is satisfied for all n. Define $T_n = \sup\,(0 \leqslant k \leqslant n \colon H_k = 0)$: this is the last time when we restart our game strategy and it is not a stopping time. At the time T_n we have $X_{T_n} < a$, $U_n(a,b) = U_{T_n}(a,b)$ and $W_n - W_{T_n} = X_n - X_{T_n}$ since $H_k = 1$ for all $T_n \leqslant k \leqslant n$. Now $W_{T_n} - W_0 \geqslant (b-a)\, U_{T_n}(a,b)$ since every upcrossing make us gain at least $(b-a)$. Therefore

$$W_n - W_0 = W_{T_n} - W_0 + X_n - X_{T_n} \geqslant (b-a)\, U_{T_n}(a,b) + X_n - a$$

$$= (b-a)\, U_n(a,b) + (X_n - a)_+ - (X_n - a)_-$$

$$\geqslant (b-a)\, U_n(a,b) - (X_n - a)_-\,.$$

From the fact that $(H_n)_{n\geqslant 1}$ is previsible and from $0 \leqslant H_n$ we have that $(W_n)_{n\geqslant 1}$ is a super-martingale:

$$0 \geqslant \mathbb{E}[W_n - W_0] \geqslant \mathbb{E}[U_n(a,b)](b-a) - \mathbb{E}[(X_n - a)_-]$$

(recall: in a unfavorable game, no strategy can allow to win in average). We deduce the following lemma (since $\mathbb{E}[(X_n - a)_-] \leqslant \mathbb{E}[(X_n - a)_-] + \mathbb{E}[(X_n - a)_+] = \mathbb{E}[|X_n - a|]$)

Lemma 1. *(Doob's upcrossing inequality) For all $a < b$ and $n \geqslant 1$ we have that*

$$\mathbb{E}[U_n(a,b)] \leqslant \frac{\mathbb{E}[|X_n - a|]}{b-a}.$$

This gives an estimate of the number of upcrossing of the interval $[a,b]$ by the process $(X_k)_{1\leqslant k\leqslant n}$ as a function of an average over its terminal value. An important consequence for super-martingales uniformly bounded in L^1 is the following corollary:

Corollary 2. *Let $(X_n)_{n\geqslant 1}$ be a super-martingale uniformly bounded in L^1(i.e. $\sup_n \mathbb{E}[|X_n|] < +\infty$). Then if we note $U(a,b) = \sup_{n\geqslant 1} U_n(a,b)$ the number of upcrossings of $[a,b]$ by $(X_n)_{n\geqslant 1}$, we have*

$$\mathbb{P}(U(a,b) = +\infty) = 0,$$

for all $a < b$.

Proof. By Doob's upcrossing inequality we have

$$\mathbb{E}[U_n(a,b)] \leqslant \frac{a + \mathbb{E}[|X_n|]}{b-a} \leqslant \frac{a + \sup_n \mathbb{E}[|X_n|]}{b-a} < +\infty$$

for all $a<b$ and $n\geqslant 1$. By monotone convergence,

$$\mathbb{E}[U(a,b)] = \lim_{n\to\infty}\mathbb{E}[U_n(a,b)] \leqslant \frac{a+\sup_n\mathbb{E}[|X_n|]}{b-a} < +\infty$$

and therefore $\mathbb{P}(U(a,b)=+\infty)=0$ for all $a<b$. □

This shows that a super-martingale uniformly bounded in L^1 cannot oscillate too much and that this is linked to the impossibility to find winning strategies on such a super-martingale. Reciprocally a similar theorem can show that a sub-martingale uniformly bounded in L^1 does not allow an infinity of downcrossings so that playing over it we cannot loose an unbounded amount of money.

Theorem 3. *(Doob's (sub-)martingale convergence theorem) A sub-martingale $(X_n)_{n\geqslant 1}$ bounded in L^1 converges a.s. towards a limit $X_{\infty}\in L^1$.*

Proof. The process $(Y_n=-X_n)_n$ is a super-martingale bounded in L^1. Let $L_+=\limsup_n Y_n$ and $L_-=\liminf_n Y_n$. Assume that $\mathbb{P}(L_-<L_+)>0$ (i.e. $(Y_n)_n$ does not converge a.s.). By continuity of the probability $\mathbb{P}$ there exist $a<b$ such that $\mathbb{P}(L_-<a<b<L_+)>0$. Now

$$\{L_-<a<b<L_+\}\subseteq\{U(a,b)=+\infty\}$$

and we obtain that $\mathbb{P}(U(a,b)=+\infty)>0$ in contradiction with the consequences of the uniform L^1 boundedness of $(Y_n)_{n\geqslant 1}$. We must therefore have $\mathbb{P}(L_-<L_+)=0$ which gives the almost sure convergence of $(Y_n)_n$ towards $Y_\infty=L_-=L_+$ and therefore of $(X_n)_{n\geqslant 1}$ towards $X_\infty=-Y_\infty$. By Fatou's lemma $\mathbb{E}[|X_\infty|]=\mathbb{E}[\liminf_n|X_n|]\leqslant\liminf_n\mathbb{E}[|X_n|]<+\infty$ which gives that $X_\infty\in L^1$. □

Of course the above theorem is also true for supermartingales and for martingales. Instead of requiring boundedness in L^1 we could only ask for one-side a.s. boundedness, provided the process has the right behavior, in particular positive processes which are supermartingales converge as claimed in the following theorem.

Theorem 4. *(Doob's super-martingale convergence theorem) A positive super-martingale $(X_n)_{n\geqslant 1}$ converges a.s. towards a (positive) limit $X_\infty\in L^1$.*

Proof. We have that $\mathbb{E}[|X_n|]=\mathbb{E}[X_n]\leqslant\mathbb{E}[X_0]$ by positivity and by the super-martingale property. Therefore $(-X_n)_{n\geqslant 1}$ is a sub-martingale uniformly bounded in L^1. By the previous theorem, it converges towards a limit $-X_\infty\in L^1$. □

Remark 5. Note that even if a sub-martingale uniformly bounded in L^1 converges a.s. to a limit which is still in L^1, the convergence do not always takes place in L^1. Here's a counterexample. Let $(Z_n)_{n\geqslant 0}$ an i.i.d. sequence with $\mathbb{P}(Z_n=+1)=1-\mathbb{P}(Z_n=-1)=p$ and let $u>1$, $X_0=x>0$ and $X_{n+1}=u^{Z_{n+1}}X_n$. Assume that $p=1/(1+u)$ in such a way that $\mathbb{E}[u^{Z_{n+1}}]=1$. Then it is easy to verify that $(X_n)_{n\geqslant 0}$ is a positive martingale and $\mathbb{E}[X_n]=\mathbb{E}[X_0]=x$. By the strong law of large numbers

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}Z_k=\mathbb{E}[Z_1]=2p-1=\frac{1-u}{1+u}<0,$$

and therefore

$$\left(\frac{X_n}{x}\right)^{1/n} \to u^{2p-1} < 1 \qquad a.s.$$

From this we conclude that $X_n \to 0$ a.s., but we already seen that $\mathbb{E}[X_n] = x > 0$. We conclude that $X_n \nrightarrow 0$ in L^1.

2 Martingales bounded in L^2

Let us start to investigate some additional condition under which we have also convergence in L^1 of the martingale, instead that just almost sure convergence.

Theorem 6. *Let $(M_n)_{n\geqslant 0}$ a martingale bounded in L^2, i.e. such that $\alpha = \sup_{n\geqslant 0}\mathbb{E}[M_n^2] < +\infty$. Then the sequence $(M_n)_n$ converges in L^2 and a.s.. to a r.v. $M_\infty \in L^2$ and moreover*

$$M_n = \mathbb{E}[M_\infty|\mathscr{F}_n]$$

for all $n \geqslant 0$.

Proof. We write the martingale as sum of its increments: $M_n = M_0 + \sum_{k=1}^n \Delta M_k$ and we remark that the increments are orthogonal: if $n > k$,

$$\mathbb{E}[\Delta M_n \Delta M_k] = \mathbb{E}[\mathbb{E}[\Delta M_n \Delta M_k|\mathscr{F}_{n-1}]] = \mathbb{E}[\mathbb{E}[\Delta M_n|\mathscr{F}_{n-1}]\Delta M_k] = 0$$

since $\Delta M_k \hat{\in} \mathscr{F}_k \subseteq \mathscr{F}_{n-1}$. Therefore

$$\mathbb{E}[M_n^2] = \mathbb{E}[M_0^2] + \sum_{k=1}^n \mathbb{E}[(\Delta M_k)^2]$$

which implies that the sequence $(\mathbb{E}[M_n^2])_n$ is increasing and that

$$\mathbb{E}[M_0^2] + \sum_{k=1}^\infty \mathbb{E}[(\Delta M_k)^2] = \sup_n \mathbb{E}[M_n^2] = \alpha.$$

Moreover, by a similar computation, for all $k' \geqslant k \geqslant n$,

$$\mathbb{E}[|M_{k'} - M_k|^2] = \sum_{\ell=k+1}^{k'} \mathbb{E}[(\Delta M_\ell)^2] \leqslant \sum_{\ell=n+1}^{\infty} \mathbb{E}[(\Delta M_\ell)^2] \to 0$$

when $n \to +\infty$. From which we deduce that the sequence $(M_n)_{n\geqslant 0}$ is Cauchy in L^2. Let $M_\infty = \lim_n M_n$ in L^2. Given that the martingale is also bounded in $L^1 \subseteq L^2$ then $M_n \to X$ a.s. We want also to show that $M_\infty = X$ a.s. By the L^2 convergence of M_n towards M_∞ we can deduce that there exists a subsequence $(n_k)_{k\geqslant 1}$ such that M_{n_k} converges a.s. towards M_∞. But then $M_\infty = \lim_k M_{n_k} = \lim_n M_n = X$ a.s..

Now for all $m \geqslant n$ we have $M_n = \mathbb{E}[M_m|\mathscr{F}_n]$ and by the contractivity in L^2 of the conditional expectation:

$$\|\mathbb{E}[M_\infty|\mathscr{F}_n] - M_n\|_2 = \|\mathbb{E}[M_\infty|\mathscr{F}_n] - \mathbb{E}[M_m|\mathscr{F}_n]\|_2 \leqslant \|M_m - M_\infty\|_2$$

which tends to 0 as $m \to \infty$. Therefore $\|\mathbb{E}[M_\infty|\mathscr{F}_n] - M_n\|_2 = 0$ and $M_n = \mathbb{E}[M_\infty|\mathscr{F}_n]$. □

Exercise 1. Recall the argument which leads to a.s. convergence for a subsequence of a sequence of random variables converging in L^2.

Exercise 2. Try to prove that M_∞ can be chosen to be $\mathscr{F}_\infty = \sigma(\mathscr{F}_n : n \geqslant 0)$ measurable.