Note 5

Closed martingales.

see also A. Bovier's script for SS17, Chapter 2 [pdf].

Definition 1. *A martingale $(X_n)_{n \geqslant 0}$ of the form $X_n = \mathbb{E}[Z|\mathscr{F}_n]$ for some $Z \in L^1$ is called closed. If $Z \in L^p$ for some $p \geqslant 1$ then the martingale is closed in L^p.*

A closed martingale is bounded in L^1 but we have already seen that L^1-boundedness is not enough to guarantee closedness. Moreover we have seen that martingales bounded in L^2 are closed in L^2 (and therefore in L^1). We will now investigate general tools to study martingales in L^p with $p \geqslant 1$ where we do not have at disposal the Hilbert geometry of L^2.

1 Doob's inequalities

Theorem 2. *(Doob's maximal inequality) Let $(X_n)_{n \geqslant 0}$ be a positive submartingale and let $X_n^* = \sup_{k \leqslant n} X_k$, then for all $\lambda > 0$ and all $n \geqslant 0$,*

$$\lambda \mathbb{P}(X_n^* \geqslant \lambda) \leqslant \mathbb{E}[X_n \mathbb{1}_{X_n^* \geqslant \lambda}] \leqslant \mathbb{E}[X_n].$$

Proof. Let $T = \inf \{n \geqslant 0 : X_n \geqslant \lambda\}$. Then $\{T \leqslant n\} = \{X_n^* \geqslant \lambda\}$ and $T \wedge n$ is a bounded stopping time. By positivity of X_n,

$$\mathbb{E}[X_n] = \mathbb{E}[X_n \mathbb{1}_{T \leqslant n} + X_n \mathbb{1}_{T > n}] \geqslant \mathbb{E}[X_n \mathbb{1}_{T \leqslant n}]$$

which gives the second inequality to be proven. Then we observe that $\{T \leqslant n\} \in \mathscr{F}_{T \wedge n}$ (prove it) and therefore that

$$\mathbb{E}[X_n \mathbb{1}_{T \leqslant n}] = \mathbb{E}[\mathbb{E}[X_n|\mathscr{F}_{T \wedge n}] \mathbb{1}_{T \leqslant n}].$$

Moreover by the optional stopping theorem applied to the two bounded stopping times $n \geqslant T \wedge n$ we get $\mathbb{E}[X_n|\mathscr{F}_{T \wedge n}] \geqslant X_{T \wedge n}$ and

$$\mathbb{E}[X_n \mathbb{1}_{T \leqslant n}] \geqslant \mathbb{E}[X_{T \wedge n} \mathbb{1}_{T \leqslant n}] = \mathbb{E}[X_T \mathbb{1}_{T \leqslant n}] \geqslant \lambda \mathbb{P}(T \leqslant n) = \lambda \mathbb{P}(X_n^* \geqslant \lambda),$$

where we used that on the event $\{T \leqslant n\}$ we have $X_{T \wedge n} = X_T \geqslant \lambda$. □

An interesting use of the Doob's decomposition allow to extend the maximal inequality to all super- or sub-martingales without positivity assumptions.

Corollary 3. *Let $(X_n)_{n \geqslant 0}$ be a super- or sub-martingale and let $|X|_n^* = \sup_{k \leqslant n} |X_k|$, then for all $\lambda > 0$ and all $n \geqslant 0$,*

$$\lambda \mathbb{P}(|X|_n^* \geqslant 3\lambda) \leqslant 3\mathbb{E}[|X_0|] + 4\mathbb{E}[|X_n|].$$

Proof. Left as exercise. Use Doob's decomposition to reduce the proof to Theorem 2. □

Theorem 4. *(Doob's L^p inequalities) Let $(X_n)_{n\geqslant 0}$ a martingale or a positive submartingale. Then for all $p>1$, letting $X_n^* = \sup_{k\leqslant n}|X_k|$ we have*

$$\|X_n^*\|_p \leqslant \frac{p}{p-1}\|X_n\|_p, \qquad n\geqslant 0.$$

Proof. If X is a martingale, then $(|X_n|)_{n\geqslant 0}$ is a positive submartingale, so it suffice to consider the latter. For any $L>0$ we have

$$\begin{aligned}
\mathbb{E}[(X_n^*\wedge L)^p] &= \mathbb{E}\left[\int_0^{L\wedge X_n^*} p x^{p-1}\mathrm{d}x\right]\\
&= \mathbb{E}\left[\int_0^{L} p x^{p-1}\mathbb{1}_{X_n^*\geqslant x}\mathrm{d}x\right]\\
&= \int_0^L p x^{p-1}\mathbb{P}(X_n^*\geqslant x)\mathrm{d}x \qquad \text{(via Fubini)}\\
&\leqslant \int_0^L p x^{p-2}\mathbb{E}[X_n\mathbb{1}_{X_n^*\geqslant x}]\mathrm{d}x \qquad \text{(by Theorem 2)}\\
&= \frac{p}{p-1}\mathbb{E}[X_n(X_n^*\wedge L)^{p-2}] \qquad \text{(via Fubini again)}\\
&\leqslant \frac{p}{p-1}\|X_n\|_p\|X_n^*\wedge L\|_p^{p-1} \qquad \text{(by Hölder's inequality)}
\end{aligned}$$

where in the last line we used the Hölder inequality with exponents $p, q = p/(p-1)$. From this we deduce that

$$\|X_n^*\wedge L\|_p \leqslant \frac{p}{p-1}\|X_n\|_p$$

and letting $L\to\infty$ via monotone convergence we obtain the statement. □

2 Martingales in L^p

Theorem 5. *Let $(X_n)_{n\geqslant 0}$ be a martingale and $p>1$. Then the following statements are equivalent*

a) X is bounded in L^p, i.e. $\sup_n\|X_n\|_p<\infty$;

b) X converges a.s. and in L^p;

c) There exists a random variable $X_\infty\in L^p$ such that $X_n=\mathbb{E}[X|\mathscr{F}_n]$ for all $n\geqslant 0$.

Proof. a)⇒b). Since X is bounded in L^p it is also bounded in L^1 and by the martingale convergence theorem there exists a r.v. $X_\infty\in L^1$ such that $X_n\to X_\infty$ almost surely. By Doob's inequality $X_n^*=\sup_{k\leqslant n}|X_k|$ satisfies $\|X_n^*\|_p\lesssim\|X_n\|_p$ and by monotone convergence $\|X_\infty^*\|_p\lesssim\sup_{n\geqslant 0}\|X_n\|_p$, so $|X_n-X_\infty|\leqslant 2X_\infty^*\in L^p$ and by dominated convergence $\|X_n-X_\infty\|_p\to 0$.

b)⇒c). Let $X_\infty=\lim_n X_n$ (with $X_\infty=0$ when the sequence does not converge). Let $Z_n=\mathbb{E}[X_\infty|\mathscr{F}_n]$ and consider

$$\|Z_n-X_n\|_p=\|\mathbb{E}[X_\infty|\mathscr{F}_n]-X_n\|_p=\liminf_{m\to\infty}\|\mathbb{E}[X_\infty-X_m|\mathscr{F}_n]\|_p\leqslant\liminf_{m\to\infty}\|X_\infty-X_m\|_p=0$$

so we obtain $X_n = \mathbb{E}[X_\infty | \mathscr{F}_n]$.

c)⇒a). Immediate from the conditional Jensen's inequality. □

Corollary 6. *Let $(X_n)_{n\geqslant 0}$ a martingale closed in L^p with $X_n = \mathbb{E}[Z|\mathscr{F}_n]$ for some $Z \in L^p$. Then*

$$X_n \to X_\infty = \mathbb{E}[Z|\mathscr{F}_\infty]$$

almost surely and in L^p.

Proof. By Theorem 5 we have $X_n \to X_\infty$ in L^p and almost surely with $X_n = \mathbb{E}[X_\infty|\mathscr{F}_n]$. Let $A \in \mathscr{F}_n \subseteq \mathscr{F}_\infty$ for some n. Then by definition of conditional expectation

$$\mathbb{E}[X_\infty \mathbb{1}_A] = \mathbb{E}[X_n \mathbb{1}_A] = \mathbb{E}[Z \mathbb{1}_A].$$

Therefore this equality holds also for all $A \in \cup_{n\geqslant 0}\mathscr{F}_n$ which is a π-system which generates $\mathscr{F}_\infty$, moreover the family of sets $A \in \mathscr{F}$ for which the equality is true is easily seen to be a λ-system, therefore the equality is true for $\sigma(\cup_{n\geqslant 0}\mathscr{F}_n) = \mathscr{F}_\infty$ and we deduce that $\mathbb{E}[Z|\mathscr{F}_\infty] = X_\infty$. □

3 Uniformly integrable martingales

Definition 7. *A UI martingale $(X_n)_{n\geqslant 0}$ is a martingale which is uniformly integrable.*

Theorem 8. *Let $(X_n)_{n\geqslant 0}$ be a martingale, then the following statements are equivalent:*

a) X is uniformly integrable;

b) $X_n \to X_\infty$ almost surely and in L^1;

c) there exists $Z \in L^1$ such that $X_n = \mathbb{E}[Z|\mathscr{F}_n]$ for all $n \geqslant 0$, (i.e. X is closed in L^1).

Proof. a)⇒b). From UI we deduce that $(X_n)_{n\geqslant 1}$ is bounded in L^1 and by the martingale convergence theorem that, almost surely $X_n \to X_\infty \in L^1$. However from UI and almost sure convergence we deduce that $X_n \to X_\infty$ also in L^1.

b)⇒c). Write $X_n = \mathbb{E}[X_m|\mathscr{F}_n]$ for all $m \geqslant n$. Since $X_m \to X_\infty$ in L^1 we have

$$\|X_n - \mathbb{E}[X_\infty|\mathscr{F}_n]\|_1 = \lim_{m\to\infty} \|\mathbb{E}[X_m|\mathscr{F}_n] - \mathbb{E}[X_\infty|\mathscr{F}_n]\|_1$$

$$\leqslant \liminf_{m\to\infty} \|\mathbb{E}[X_m - X_\infty|\mathscr{F}_n]\|_1 \leqslant \liminf_{m\to\infty} \|X_m - X_\infty\|_1 = 0$$

and as a consequence $X_n = \mathbb{E}[X_\infty|\mathscr{F}_n]$ with $X_\infty \in L^1$.

c)⇒a). We already have shown that the family $(\mathbb{E}[Z|\mathscr{G}])_{\mathscr{G}}$ is uniformly integrable if $Z \in L^1$, therefore also the family $(X_n = \mathbb{E}[Z|\mathscr{F}_n])_n$ is UI and is indeed a martingale. □

Remark 9. If $(X_n)_n$ is a UI supermartingale (resp. submartingale) then $X_n \to X_\infty$ almost surely and in L^1 and $\mathbb{E}[X_\infty|\mathscr{F}_n] \leqslant X_n$ (resp. $\geqslant$) for all $n \geqslant 0$.

If X is an UI martingale, then is natural to define

$$X_T = \sum_{n \geqslant 0} X_n \mathbb{1}_{T=n} + X_\infty \mathbb{1}_{T=\infty}$$

for all stopping times T (not necessarily finite), where $X_\infty = \lim_n X_n$ and $X_n = \mathbb{E}[X_\infty | \mathscr{F}_n]$. The the following extension of the optional stopping theorem is true.

Theorem 10. *(Optional stopping for UI martingales) Let X be a UI martingale and $S \leqslant T$ two stopping times. Then*

$$\mathbb{E}[X_T | \mathscr{F}_S] = X_S.$$

Proof. Note that $|X_T| = \sum_{n \geqslant 0} |X_n| \mathbb{1}_{T=n} + |X_\infty| \mathbb{1}_{T=\infty}$ and

$$\mathbb{E}[|X_T|] \leqslant \sum_{n \geqslant 0} \mathbb{E}[|\mathbb{E}[X_\infty | \mathscr{F}_n]| \mathbb{1}_{T=n}] + \mathbb{E}[|X_\infty| \mathbb{1}_{T=\infty}]$$

$$\leqslant \sum_{n \geqslant 0} \mathbb{E}[|X_\infty| \mathbb{1}_{T=n}] + \mathbb{E}[|X_\infty| \mathbb{1}_{T=\infty}] \leqslant \mathbb{E}[|X_\infty|].$$

Moreover for $A \in \mathscr{F}_T$ we have

$$\mathbb{E}[X_\infty \mathbb{1}_A] = \sum_{n \geqslant 0} \mathbb{E}[X_\infty \mathbb{1}_{A \cap \{T=n\}}] + \mathbb{E}[X_\infty \mathbb{1}_{A \cap \{T=\infty\}}]$$

$$= \sum_{n \geqslant 0} \mathbb{E}[X_n \mathbb{1}_{A \cap \{T=n\}}] + \mathbb{E}[X_\infty \mathbb{1}_{A \cap \{T=\infty\}}] = \mathbb{E}[X_T \mathbb{1}_A]$$

therefore $\mathbb{E}[X_\infty | \mathscr{F}_T] = X_T$ and by the tower property we can conclude. □