Note 7

Optimal stopping with finite horizon

The problem we would like to analyse in this part is the following. Consider an adapted process $(Y_n)_{n\geqslant 1}$ and try to optimize the value of $\mathbb{E}[Y_T]$ among all possible stopping times T for the given filtration. One could interpret this situation as a game. We imagine that $Y_n(\omega)$ is the gain which we obtain if we decide to stop at time n and that we try to find a stopping strategy to maximize the average gain. Stopping times are of course the natural class of admissible stopping strategies.

We will consider only problems in finite horizon, namely we fix $N\in\mathbb{N}$, we let $\mathcal{T}_N$ the set of all stopping times bounded by N and we look for the optimal average gain J_N with horizon N:

$$J_N = \sup_{T\in\mathcal{T}_N} \mathbb{E}[Y_T].$$

We say that $T^\star \leqslant N$ is an optimal stopping time if $\mathbb{E}[Y_{T^\star}] = J_N$.

Notation: We let $\inf_N A = \inf A$ for all $A\subseteq\mathbb{R}$ with $A\neq\emptyset$ and $\inf_N\emptyset = N$.

As with many optimisation problems, the solution of the optimal stopping problem above goes via the determination of a suitable *value function* $(Z_n)_n$ associated to the choices still available at time n. The value function represents the average gain conditional on the information gained up to time n, namely conditionally on $\mathcal{F}_n$. It must satisfy the following properties:

a) $(Z_n)_n$ is an adapted process. We must be able to determine it only as a function of the information available at time n.

b) $Z_n\geqslant Y_n$: at time n what I hope to gain cannot be less that what I would gain stopping immediately at n.

c) $Z_n\geqslant\mathbb{E}[Z_{n+1}|\mathcal{F}_n]$: my current position has a value which cannot be inferior to what I expect to gain if I would continue one step further (given that I already know $\mathcal{F}_n$).

Indeed at each step $n<N$ I have two options: either stop or continue. At the final step N I do not have anymore the option to continue, I must stop and gain Y_N. Therefore $Z_N = Y_N$ and we can define a value function by the backward equation:

$$Z_N = Y_N, \quad Z_n = \sup(Y_n, \mathbb{E}[Z_{n+1}|\mathcal{F}_n]) \quad \text{pour } 1\leqslant n<N \tag{1}$$

From this definition we see that Z is a supermartingale which bound from above Y. In particular we will show that it is the *Snell's envelope of* Y, i.e. the smallest supermartingale Q such that $Y_n\leqslant Q_n$ for all $0\leqslant n\leqslant N$.

Theorem 1. *Let $(Y_n)_{n\geqslant 1}$ be an adapted process such that $\mathbb{E}|Y_n|<\infty$ pour tout $n\geqslant 1$. Define $(Z_n)_n$ by eq.(1) and let $T^\star = \inf\{k\leqslant N: Y_k = Z_k\}$. Then the sequence $(Z_{n\wedge T^\star})_{n\geqslant 1}$ is a martingale and*

$$\mathbb{E}[Z_1] = \mathbb{E}[Z_{T^\star}] = \mathbb{E}[Y_{T^\star}] = J_N.$$

The stopping time $T^\star$ is optimal and Z is the Snell envelope of Y.

Proof. By definition $Z_n \geqslant \mathbb{E}[Z_{n+1}|\mathscr{F}_n]$ and $Z_n \geqslant Y_n$. On the event $\{T^\star \geqslant n\}$ we have $Z_n = \mathbb{E}[Z_{n+1}|\mathscr{F}_n]$, therefore the process $(\tilde{Z}_n := Z_{n\wedge T^\star})_n$ is a martingale wrt. $(\mathscr{F}_n)_{n\geqslant 1}$. Indeed $\mathbb{E}[1_A Z_{(n+1)\wedge T^\star}] = \mathbb{E}[1_A Z_{n\wedge T^\star}]$ for all $A \in \mathscr{F}_n$. As a consequence, if we consider the two stopping times $n \wedge T^\star$ and $T^\star$, we have $n \wedge T^\star \leqslant T^\star$ and $\mathbb{E}[\tilde{Z}_{T^\star}|\mathscr{F}_{n\wedge T^\star}] = \tilde{Z}_{n\wedge T^\star}$ which implies $\mathbb{E}[Z_{T^\star}|\mathscr{F}_{n\wedge T^\star}] = Z_{n\wedge T^\star}$. Taking the expectation of this last equation, we have, for all $T \leqslant N$:

$$\mathbb{E}[Y_T] \leqslant_{(1)} \mathbb{E}[Z_T] \leqslant_{(2)} \mathbb{E}[Z_1] =_{(3)} \mathbb{E}[Z_{T^\star}] =_{(4)} \mathbb{E}[Y_{T^\star}]$$

where the bound (1) is due to the fact that $Y_n \leqslant Z_n$ for all $n \in [0, N]$ and therefore for all stopping time $T \leqslant N$. The bound (2) is the supermartingale property of Z, the equality (3) is due to the martingale property of the stopped process $\tilde{Z}_n$ and finally the equality (4) is due to the fact that $Y_{T^\star} = Z_{T^\star}$ as a consequence of the definition of $T^\star$. Since this is true for any stopping time $T \leqslant N$ we have that $\mathbb{E}[Y_{T^\star}] = J_N$ and therefore that $T^\star$ is an optimal stopping time for Y. The optimal gain is given by $J_N = \mathbb{E}[Z_1]$. We show now that Z is the Snell envelope of Y. Indeed let Q another supermartingale which bounds from above Y: at the final time we need to have $Q_N \geqslant Y_N = Z_N$. Moreover if we have $Q_n \geqslant Z_n$ for all n such that $N \geqslant n > k$ then $Q_k \geqslant \mathbb{E}[Q_{k+1}|\mathscr{F}_k] \geqslant \mathbb{E}[Z_{k+1}|\mathscr{F}_k]$ and $Q_k \geqslant Y_k$, therefore we have also $Q_k \geqslant Z_k$ and we establish the domination also at time k. By a backward induction we have domination for all $1 \leqslant k \leqslant N$ and as a consequence Z is indeed the smallest supermartingale above Y. □

Corollary 2. *The stopping time $T^\star$ is the smallest optimal stopping time: if S is another optimal stopping time, then $T^\star \leqslant S$ almost surely.*

Proof. Assume that $\mathbb{P}(T^\star > S) > 0$. Then for $\omega \in \Omega$ such that $T^\star(\omega) > S(\omega)$ we have $Y_S(\omega) < Z_S(\omega)$ since $T^\star(\omega)$ is the first k for which $Y_k(\omega) = Z_k(\omega)$. Given that the event $\{T^\star > S\}$ has a positive probability, on obtain that $\mathbb{E}[Y_S] < \mathbb{E}[Z_S]$ striclty. But, by the supermartingale property of Z, we deduce that $\mathbb{E}[Y_S] < \mathbb{E}[Z_S] \leqslant \mathbb{E}[Z_1] = J_N$ and this is in contradiction with the hypothesis that S is optimal (i.e. $\mathbb{E}[Y_S] = \sup_T \mathbb{E}[Y_T] = J_N$). □

Remark 3. Observe that an equivalent definition of $T^\star$ is

$$T^\star = \inf\{k \leqslant N : Y_k \geqslant \mathbb{E}[Z_{k+1}|\mathscr{F}_k]\}.$$

Corollary 4. *The stopping time $T^\sharp = \inf\{k \leqslant N : Y_k > \mathbb{E}[Z_{k+1}|\mathscr{F}_k]\}$ is the largest optimal stopping time: if S is an optimal stopping time, then $S \leqslant T^\sharp$ almost surely.*

Proof. Assume that $\mathbb{P}(T^\sharp < S) > 0$. We note that $\tilde{Z}_n = Z_{n\wedge(T^\sharp+1)}$ is a martingale (indeed if $n \leqslant T^\sharp$ then $Y_n \leqslant \mathbb{E}[Z_{n+1}|\mathscr{F}_n]$ and therefore $Z_n = \mathbb{E}[Z_{n+1}|\mathscr{F}_n]$). On the one hand we have $\mathbb{E}[\tilde{Z}_S|\mathscr{F}_S] = \tilde{Z}_S$ due to the martingale property of $\tilde{Z}$. We note that $\{T^\sharp \geqslant S\} \in \mathscr{F}_S$ and therefore that

$$Y_S \mathbb{1}_{T^\sharp \geqslant S} \leqslant Z_S \mathbb{1}_{T^\sharp \geqslant S} = \tilde{Z}_S \mathbb{1}_{T^\sharp \geqslant S} = \mathbb{E}[\tilde{Z}_{T^\sharp} \mathbb{1}_{T^\sharp \geqslant S}|\mathscr{F}_S] = \mathbb{E}[Z_{T^\sharp} \mathbb{1}_{T^\sharp \geqslant S}|\mathscr{F}_S]. \qquad (2)$$

On the other hand, if we let $Z_{N+1} = Z_N$ then $(Z_n)_{n=1,\ldots,N+1}$ is still a supermartingale and therefore $\mathbb{E}[Z_{S\vee(T^\sharp+1)}|\mathscr{F}_{T^\sharp+1}] \leqslant Z_{T^\sharp+1}$ (by the supermartingale inequality with the two stopping times $T^\sharp + 1 \leqslant S \vee (T^\sharp + 1) \leqslant N + 1$). From $\{T^\sharp < S\} \in \mathscr{F}_{T^\sharp}$ and $Y_S \leqslant Z_S$ we have

$$\begin{aligned}\mathbb{E}[Y_S \mathbb{1}_{T^\sharp<S}] &\leqslant \mathbb{E}[Z_S \mathbb{1}_{T^\sharp<S}] = \mathbb{E}[Z_{S\vee(T^\sharp+1)} \mathbb{1}_{T^\sharp<S}] = \mathbb{E}[\mathbb{E}[Z_{S\vee(T^\sharp+1)}|\mathscr{F}_{T^\sharp+1}] \mathbb{1}_{T^\sharp<S}] \\ &\leqslant \mathbb{E}[Z_{T^\sharp+1} \mathbb{1}_{T^\sharp<S}] = \mathbb{E}[\mathbb{E}[Z_{T^\sharp+1}|\mathscr{F}_{T^\sharp}] \mathbb{1}_{T^\sharp<S}] < \mathbb{E}[Y_{T^\sharp} \mathbb{1}_{T^\sharp<S}] \leqslant \mathbb{E}[Z_{T^\sharp} \mathbb{1}_{T^\sharp<S}].\end{aligned} \qquad (3)$$

Here we used the fact that, by the definition of $T^\sharp$ we have $Y_{T^\sharp} > \mathbb{E}[Z_{T^\sharp+1}|\mathscr{F}_{T^\sharp}]$. Eq. (2) and eq. (3) give that

$$\mathbb{E}[Y_S] = \mathbb{E}[Y_S \mathbb{1}_{T^\sharp \geqslant S}] + \mathbb{E}[Y_S \mathbb{1}_{T^\sharp < S}] < \mathbb{E}[Z_{T^\sharp} \mathbb{1}_{T^\sharp \geqslant S}] + \mathbb{E}[Z_{T^\sharp} \mathbb{1}_{T^\sharp < S}] = \mathbb{E}[Z_{T^\sharp}] = \mathbb{E}[Y_{T^\sharp}]$$

which is in contradition with the hypothesis of the optimality of S. □

Remark 5. Give a detailed proof of $Y_{T^\sharp} > \mathbb{E}[Z_{T^\sharp+1}|\mathscr{F}_{T^\sharp}]$. Start by showing that if F is an integrable random variable and T is a stopping time, then $\mathbb{E}[F|\mathscr{F}_T]\mathbb{1}_{T=n} = \mathbb{E}[F|\mathscr{F}_n]\mathbb{1}_{T=n} = \mathbb{E}[F\mathbb{1}_{T=n}|\mathscr{F}_n]$. Then write $Y_{T^\sharp} = \sum_{n=1}^N Y_n \mathbb{1}_{T^\sharp=n}$ and close the argument.

1 Markovian problems

In many optimal stopping problems the following assumptions are satisfied: there exists an adapted process $(X_n)_{n\geqslant 0}$ taking values in a measure space $(E, \mathscr{E})$ for which

a) For any bounded measurable function $f: E \to \mathbb{R}$ we have, for any $n \geqslant 0$

$$\mathbb{E}[f(X_{n+1})|\mathscr{F}_n] = \mathbb{E}[f(X_{n+1})|X_n] = (P_{n+1}f)(X_n)$$

for some probability kernel $P_{n+1}: E \times \mathscr{E} \to [0,1]$.

b) The gain Y_n can be expressed as $Y_n = \varphi_n(X_n)$ for some measurable $\varphi_n: E \to \mathbb{R}$.

Then is not difficult to prove that the Snell envelope $(Z_n)_{n\in\{1,\ldots,N\}}$ of $(Y_n)_{n\in\{1,\ldots,N\}}$ has the form $Z_n = V_n(X_n)$ where $(V_n: E \to \mathbb{R})_{n\in\{1,\ldots,N\}}$ is the sequence of functions given by the backwards recurrence:

$$\begin{cases} V_N = \varphi_N \\ V_n = \sup(\varphi_n, (P_{n+1}V_{n+1})) \quad n \in \{1,\ldots,N-1\} \end{cases}$$

here $(P_{n+1}f)(x) := \int_E f(y) P_{n+1}(x, \mathrm{d}y)$ denotes the action of the kernel P_{n+1} on the function f by integration.

Exercise 1. Prove it.