V3F1 Stochastic Processes – Problem Sheet 10

Distributed June 7th, 2019. At most in groups of 2. Solutions have to be handed in before noon on Friday June 21st into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [3 pts] Let $(E,\mathscr{E})$ be a discrete space endowed with the full σ-algebra of its subsets. Prove that any Markov chain on E can be realised as a random recurrence. Namely that there is a random recurrence which has the same law as the given Markov chain.

Exercise 2. [1+1+1+1 pts] Consider throwing a dice repeatedly and determine whether the following processes are Markov chains and in that case describe their transition kernel.

a) At the n-th throws, X_n is the biggest result obtained.

b) At the n-th throws, N_n is the number of 6s obtained.

c) At the n-th throws, C_n is the number of dice thrown after the last 6 observed (e.g. $C_n=0$ if we observe 6 at time n and $C_n=4$ if the last numbers we observed are $\cdots,6,3,4,2,2$).

d) $B_n=\sum_{k=0}^n N_k$.

Exercise 3. [1+1+2 pts] Let $(X_n)_{n\geqslant 0}$ be an homogeneous Markov chain on $(E,\mathscr{E})$ with transition kernel P: $E\times\mathscr{E}\to[0,1]$. Determine whether the following processes are Markov chains and in that case describe their transition kernel.

a) $W_n=X_{n+k}$, $n\geqslant 0$, for some fixed $k\geqslant 1$.

b) $Y_n=X_{2n}$, $n\geqslant 0$.

c) $Z_n=X_{T_n}$, $n\geqslant 0$. Here $T_0=0$ and $T_n=S_1+\cdots+S_n$ where $(S_n)_{n\geqslant 1}$ is a sequence of i.i.d. random variables taking values in $\mathbb{N}+1$ and independent of $(X_n)_{n\geqslant 0}$.

Exercise 4. (Feynman–Kac formula) [2+3 pts] Let $(X_n)_n$ be a Markov chain on $(E,\mathscr{E})$ with generator $\mathscr{L}$. Let $w:E\to\mathbb{R}_+$ a non-negative function:

a) For which function v is the process

$$M_n=\exp[-\sum_{k=0}^{n-1} w(X_k)]v(X_n), \qquad n\geqslant 0,$$

a martingale?

b) Let $D\in\mathscr{E}$ such that $T=T_{D^c}=\inf\{n\geqslant 0: X_n\in D^c\}<\infty$ $\mathbb{P}_x$-a.s. for all $x\in E$ and let v be a bounded solution of the boundary value problem

$$\begin{cases} \mathscr{L}v(x) &= (e^{w(x)}-1)v(x), & x\in D \\ v(x) &= f(x), & x\in D^c. \end{cases}$$

where $f:\mathscr{E}\to\mathbb{R}$ is a bounded function. Show using (a) that

$$v(x)=\mathbb{E}_x\left[\exp\left(-\sum_{k=0}^{T-1} w(X_k)\right)f(X_T)\right].$$

Exercise 5. [2+2+2 pts] Let $(X_n)_n$ be a Markov chain on a countable state space E, realised on the canonical space with shift $(\theta_n)_n$. Let $\mathbb{P}_x$ the law of the Markov chain with $X_0 = x$ a.s. For $A \subseteq E$ and $T_A^+ := \inf\{n > 0 : X_n \in A\}$, $T_A^{n+1} = T_A^+ \circ \theta_{T_A^n}$ the return times to A and for $x \in E$ let $T_x^n = T_{\{x\}}^n$. Assume that all the stopping times considered here are almost surely finite.

a) Fix $x \in E$ and prove that the interarrival times $(\tau_x^n := T_x^n - T_x^{n-1})_{n \geqslant 1}$ at x are i.i.d.

b) Show that the number $N_x = \#\{n \geqslant 0 : X_n = x\}$ of visits of x is almost surely infinite.

c) Fix $A \subseteq E$ and prove that for any $x \in A$, the process $(Y_n = X_{T_A^n})_{n \geqslant 0}$ under $\mathbb{P}_x$ is a Markov chain on the state space A. Compute its transition kernel.

Exercise 6. [1+2+2+2+1 pts] Two players A and B bet repeatedly €1 against each other. Each time the probability that A wins is $p \in (0,1)$ and successive games are independent. Let X_n be the wealth of A after n games. We assume that their total wealth is $L \geqslant 1$. Therefore if initially $X_0 = x \in \{0, \ldots, L\}$ then the wealth of B is $L - x$. The game ends when one of the two player goes broke, so when $X_n = 0$ or $X_n = L$. Let $T = \inf\{n \geqslant 0 : X_n \in \{0, L\}\}$ be such a random time. Let $u(x)$ be the probability that A wins given that $X_0 = x$ (and B starts with $L - x$), i.e. $u(x) = \mathbb{P}(X_T = L, T < \infty | X_0 = x)$.

a) Show that $(X_n)_{n \geqslant 0}$ is a Markov chain and give its transition kernel

b) Show that u solves a linear equation and solve it.

c) Consider also $v(x) = \mathbb{P}(X_T = 0, T < \infty | X_0 = x)$ and deduce that $\mathbb{P}(T < \infty | X_0 = x) = 1$ for all $x \in \{0, \ldots, L\}$.

d) Let $m(x) = \mathbb{E}[T | X_0 = x]$ the average duration of the game when the player A starts at x. Find an equation for m, show it has a unique solution and solve it.

e) Deduce that playing a fair game against the Casino ($L \to +\infty$, $X_0 = x$ fixed) one is sure to get broke but that in average this will require an infinite amount of time.