V3F1 Stochastic Processes – Problem Sheet 11

Distributed June 21st, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday June 27th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [2+2+1+2+1 pts] Consider a discrete, homogeous, irreducible and positive recurrent Markov chain with state space E and transition matrix P on its canonical space. We want to prove the following law of large numbers: for any bounded function $f: E \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k) \to \sum_{x\in E} f(x)\pi(x) \tag{1}$$

where π is the unique invariant probability for the chain and the convergence holds almost surely.

a) Consider the successive return times $(T_x^n)_{n\geqslant 1}$ for a given state $x \in E$, defined as $T_x^0 = 0$ and $T_x^n = \inf\{k > T_x^{n-1}: X_k = x\}$ for $n \geqslant 1$. Moreover let $\tau_x^n = T_x^n - T_x^{n-1}$ for $n \geqslant 1$ to be the intervals between two visits to x. Prove that τ_x^n is conditionally independent of $\mathscr{F}_{T_x^{n-1}}$.

b) Prove that $(\tau_x^n)_{n\geqslant 1}$ is a family of i.i.d. geometric random variables.

c) Prove that $\frac{1}{n}\sum_{k=1}^{n} \tau_x^k \to \mathbb{E}_x[T_x]$ almost surely.

d) Let $N_n^x = \#\{k \in \{1,\ldots,n\}: X_k = x\}$ the number of returns to x up to time n. Prove that $N_n^x \to \infty$ as $n \to \infty$ almost surely and then that

$$\frac{N_n^x}{n} \to \frac{1}{\mathbb{E}_x[T_x]}.$$

e) Deduce that eq. (1) is true for functions of the form $f_x(y) = \mathbb{1}_{x=y}$ and conclude.

Exercise 2. [1+3+2 pts] Let $(U_n)_{n\geqslant 1}$ be an i.i.d family with values on $\mathbb{Z}^d$ and characteristic function

$$\varphi_{U_1}(t) = \mathbb{E}[e^{-i2\pi\langle t,U_1\rangle}], \qquad t \in \mathbb{R}^d$$

a) Verify that

$$\mathbb{P}(U_1 = z) = \int_{[0,1)^d} \varphi_{U_1}(t) e^{i2\pi\langle t,z\rangle} \mathrm{d}t.$$

b) Let $X_0 = x \in \mathbb{Z}^d$ and $X_n = X_{n-1} + U_n$ for all $n \geqslant 1$. Assume that this Markov chain is irreducible and prove that it is recurrent iff

$$\lim_{\gamma\uparrow 1-} \int_{[0,1)^d} \mathrm{Re}\left[\frac{1}{1-\gamma\varphi_{U_1}(t)}\right] \mathrm{d}t = +\infty.$$

c) Apply the above considerations to prove that the simple random walk with $\mathbb{P}(U_1 = z) = \frac{1}{2d}\mathbb{1}_{|z|=1}$ is recurrent if $d \leqslant 2$ and transient if $d \geqslant 3$.

Exercise 3. [2+2+1+1 pts] Consider the Markov chain on $\mathbb{N}$ given by the random mechanism

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \in (0,1), \\ 0 & \text{with probability } 1-p. \end{cases}$$

a) Prove that the chain is irreducible and positive recurrent.

b) Compute the invariant probability π (Hint: use the equation $\pi = \pi P$ and look for an expression of the characteristic function of π)

c) Compute $\mathbb{E}_x[T_x]$ for all $x \geqslant 0$ where $T_x = \inf\{n \geqslant 1 : X_n = x\}$.

d) Compute for all $k \geqslant 0$ the quantity

$$\mathbb{E}_0\left[\sum_{n=0}^{T_0-1} \mathbb{1}_{X_n \geqslant k}\right].$$

[BONUS: Show that for all $x, y \geqslant 0$, $\mathbb{P}_y(X_n = x) \to \pi(x)$ as $n \to \infty$.]