V3F1 Stochastic Processes – Problem Sheet 12

Distributed June 28th, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday July 4th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [1+2+1+2+2 pts] The Wright-Fisher model describes the evolution of a population of individuals with phenotype A or B. The total population size at each generation is kept constant and is equal to N. The Wright-Fisher model is a stochastic process $(X_n)_{n\in\mathbb{N}}$ with state space $S=\{0,\ldots,N\}$, where

$$X_n = \text{number of individuals of type } A \text{ in the } n\text{th generation.}$$

Thus the number of individuals of type B is just $N-X_n$. In this model, the evolution is Markovian and is given as follows: given a population at time n, each of the descendants (population at time $n+1$) takes the type from a randomly chosen parent of the nth generation. Show that

1. (X_n) is a Markov chain and compute the transition probability.
2. (X_n) is a martingale.
3. the states 0 and N are absorbing, where a state x is absorbing if $\mathbb{P}_x(X_1=x)=1$.
4. $\mathbb{P}_x[T_{\{0,N\}}<\infty]=1$ and $\mathbb{E}_x[T_{\{0,N\}}]<\infty$ for all $x\in\{0,\ldots,N\}$.
5. Compute (using e.g. Doob's Optional Stopping Theorem) the probability that the process is absorbed in N (resp. in 0).

Exercise 2. [3+3+4+2+2 pts] The (one-dimensional) Brownian motion $(B_t)_{t\geq 0}$ is a Gaussian process with state space $\mathbb{R}$ in continuous time, with covariance $\mathrm{Cov}(B_s,B_t)=\min\{s,t\}$.

Prove the following properties

a) For any choice $0\leqslant t_1<t_2<\cdots<t_n$ the family of r.v. $(B_{t_1},B_{t_1}-B_{t_2},\ldots,B_{t_{n-1}}-B_{t_n})$ is independent and B_t-B_s has the same law as B_{t-s}.

b) For any $t>s$ we have

$$\mathbb{E}[f(B_t)|B_s]=(P_{t-s}f)(B_s),$$

where the transition kernel P_t is given by

$$P_t(x,\mathrm{d}y)=\frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t}\,\mathrm{d}y.$$

c) The process $(B_t)_t$ is a Markov process wrt. the filtration $(\mathscr{F}_t)_t$ given by $\mathscr{F}_t=\sigma(B_s: s\in[0,t])$. Here the relevant Markov property is

$$\mathbb{E}[f(B_t)|\mathscr{F}_s]=\mathbb{E}[f(B_t)|B_s]=(P_{t-s}f)(X_s).$$

d) The process $(-B_t)_{t\geq 0}$ is also a Brownian motion (*symmetry property*).

e) For any $c>0$, the process defined by

$$X_t:=\frac{1}{\sqrt{c}}B_{c\cdot t},\qquad t\geqslant 0,$$

is also a Brownian motion (*scaling invariance property*).