Institute for Applied Mathematics – SS19
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V3F1 Stochastic Processes – Problem Sheet 2

Distributed April 12th 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday April 18th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [2 Pts] Show that each σ-finite measure μ on some measurable space $(\Omega, \mathscr{F})$ has a representation of the form $\mu = \sum_{n \geqslant 0} a_n \mu_n$, where for all n, $a_n \geqslant 0$ and μ_n is a probability measure on $(\Omega, \mathscr{F})$.

Exercise 2. [2+1+1+1 Pts] Let $\mathscr{X}$ and $\mathscr{Y}$ two families of random variables.

a) Show that $\mathscr{X}$ is uniformly integrable iff $\sup_{X \in \mathscr{X}} \mathbb{E}[|X| \mathbb{1}_{|X| \geqslant K}] \to 0$ as $K \to +\infty$;

b) Show that $\mathscr{X} + \mathscr{Y} = \{X + Y : X \in \mathscr{X}, Y \in \mathscr{Y}\}$ is uniformly integrable if $\mathscr{X}, \mathscr{Y}$ are;

c) Let $g: \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(x)/x \to \infty$ as $x \to +\infty$. Show that if $\sup_{X \in \mathscr{X}} \mathbb{E}[g(|X|)] < \infty$ then $\mathscr{X}$ is uniformly integrable;

d) Show that if $\mathbb{E}[\sup_{X \in \mathscr{X}} |X|] < \infty$ then $\mathscr{X}$ is uniformly integrable.

Exercise 3. [5 Pts] Let $(\Omega, \mathscr{F})$ a measure space and $X, Y: \Omega \to \mathbb{R}$ two measurable functions. We write $\sigma(X)$ for the σ-algebra generated by the function X, namely the σ-algebra generated by the sets of the form $\{\omega \in \Omega: X(\omega) \leqslant t\}$. Prove that if Y is measurable wrt. $\sigma(X)$ then there exists a measurable function ϕ: $(\mathbb{R}, \mathscr{B}(\mathbb{R})) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that $Y = \phi(X)$.

Hint: First reduce the problem to considering bounded Y. Then apply the monotone class theorem.

Exercise 4. [3 Pts] Let $(A_n)_n$ a countable partition of Ω. Let $\mathscr{G} = \sigma(A_1, \ldots)$ the σ-algebra generated by this partition. Prove that

$$E(X|\mathscr{G})(\omega) = \sum_{j: P(A_j) > 0} \frac{E(X 1_{A_j})}{P(A_j)} 1_{A_j}(\omega).$$

Exercise 5. [5 Pts] Let (X, Y) a pair of random variables with values in $\mathbb{R}^n \times \mathbb{R}^m$ and joint density $f_{X,Y}(x, y)$. Compute the conditional probability $\mathbb{E}[g(Y)|X]$ for $g(Y) \in L^1$.