Institute for Applied Mathematics – SS19
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V3F1 Stochastic Processes – Problem Sheet 3

Distributed April 16th, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday April 25th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [1+1+1+1+1 Pts] For all $X, Y \in L^1(\mathscr{F})$ and all sub-σ-algebras $\mathscr{G}, \mathscr{H} \subseteq \mathscr{F}$, prove the following properties of the conditional expectation:

a) Linearity: $\mathbb{E}[\lambda X + \mu Y|\mathscr{G}] = \lambda \mathbb{E}[X|\mathscr{G}] + \mu \mathbb{E}[Y|\mathscr{G}]$a.s. for all $\lambda, \mu \in \mathbb{R}$;

b) Jensen's inequality: for all convex $\varphi: \mathbb{R} \to \mathbb{R}$: $\mathbb{E}[\varphi(X)|\mathscr{G}] \geqslant \varphi(\mathbb{E}[X|\mathscr{G}])$a.s. ;

c) Contractivity in L^p: $\|\mathbb{E}[X|\mathscr{G}]\|_p \leqslant \|X\|_p$;

d) Telescoping: If $\mathscr{H}$ is a sub-σ-algebra of $\mathscr{G}$ then $\mathbb{E}[\mathbb{E}[X|\mathscr{G}]|\mathscr{H}] = \mathbb{E}[X|\mathscr{H}] = \mathbb{E}[\mathbb{E}[X|\mathscr{H}]|\mathscr{G}]$ a.s.;

e) If Z is $\mathscr{G}$ – measurable, $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|XZ|] < +\infty$ then $\mathbb{E}[XZ|\mathscr{G}] = Z\,\mathbb{E}[X|\mathscr{G}]$ a.s.

Exercise 2. [2 Pts] Show that, if $X_1 = X_2$ on $B \in \mathscr{G}$ (i.e. $X_1(\omega) = X_2(\omega)$ if $\omega \in B$), then $\mathbb{E}[X_1|\mathscr{G}] = \mathbb{E}[X_2|\mathscr{G}]$ on $B \in \mathscr{G}$.

Exercise 3. [2+2 Pts]

a) [Bayes' formula] Define $\mathbb{P}(A|\mathscr{G}) = \mathbb{E}[\mathbb{1}_A|\mathscr{G}]$. Show that if $\mathscr{G}$ is a sub-σ-algebra of $\mathscr{F}$ and $A \in \mathscr{F}$ then

$$\mathbb{P}(G|A) = \frac{\mathbb{E}[\mathbb{P}(A|\mathscr{G})1_G]}{\mathbb{E}[\mathbb{P}(A|\mathscr{G})]}.$$

b) Give an example with $\Omega = \{a, b, c\}$ to show that in general

$$\mathbb{E}[\mathbb{E}[X|\mathscr{F}_1]|\mathscr{F}_2] \neq \mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{F}_1].$$

Exercise 4. [2+2 Pts]

a) Let X_1 and X_2 two independent r.v. both with law Poisson(λ) with $\lambda > 0$. Let $Y = X_1 + X_2$. Compute $\mathbb{P}(X_1 = k|Y)$ for $k \geqslant 0$.

b) Let X_1 and X_2 be independent r.v. such that $\mathbb{P}(X_i > t) = e^{-t}$ for $t \geqslant 0$. Let $Y = X_1 + X_2$. Compute $\mathbb{E}[X_1|Y]$ and $\mathbb{P}(X_1 < 3|Y)$

Exercise 5. [2+3 Pts]

a) Assume that two r.v. X, Y satisfy $\mathbb{E}[Y|\mathscr{G}] = X$ for some σ-algebra $\mathscr{G} \subseteq \mathscr{F}$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2] < \infty$. Deduce that $X = Y$ a.s.

b) Prove the conditional Cauchy-Schwarz inequality:

$$\mathbb{E}[|XY||\mathscr{G}]^2 \leqslant \mathbb{E}[|X|^2|\mathscr{G}]\,\mathbb{E}[|Y|^2|\mathscr{G}].$$