V3F1 Stochastic Processes – Problem Sheet 4

Distributed April 26th, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday May 2nd into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [5 Pts] Let $(X_0, X_1, \ldots, X_n)$ be a Gaussian vector with zero mean and covariance $\Gamma = (\Gamma_{ij})_{i,j=1,\ldots,n}$. Show that

$$\mathbb{E}[X_0|X_1, \ldots, X_n] = \sum_{i=1}^{n} \lambda_i X_i \quad a.s.$$

and determine the weights $(\lambda_i)_i$ as functions of Γ. (Sugg: show that $X_0 = \sum_{i=1}^{n} \lambda_i X_i + Z$ where Z is a r.v. independent of $X_1, \ldots, X_n$, you will need to use the properties of Gaussian vectors, justify carefully your answer).

Exercise 2. [1+1+1+1+1 Pts] Let $(\mathscr{F}_n)_{n\geqslant 0}$ be a filtration, all the stopping times below are considered with respect to this filtration. Recall the convention $\inf(\emptyset) = +\infty$. Below S, T are two stopping times and $(X_n)_n$ an adapted process.

a) Show that $T \wedge S = \min(T, S)$, $T \vee S = \max(T, S)$ and $T + S$ are also stopping times.

b) Let $A \in \mathscr{B}(\mathbb{R})$. Show that $T_A = \inf\{n \geqslant 0 : X_n \in A\}$ is a stopping time.

c) Show that $(X_{n\wedge T})_n$ is an adapted process.

d) Let $(T_n)_{n\geqslant 1}$ be a sequence of stopping times. Show that $T = \inf_{n\geqslant 1} T_n$ is also a stopping time.

e) Let $(X_n)_{n\geqslant 0}$ be an adapted process and $Y_n = \max_{0\leqslant k\leqslant n} X_n$. Show that $T = \inf\{n \geqslant 1: X_n \geqslant Y_{n-1}\}$ is a stopping time and that $Y_T = X_T$ on the event $\{T < \infty\}$.

Exercise 3. [2+1+1+1+2+1+2 Pts] Let T a stopping time for the filtration $(\mathscr{F}_n)_{n\geqslant 0}$ and consider the family of events

$$\mathscr{F}_T := \{A \in \mathscr{F} : A \cap \{T \leqslant n\} \in \mathscr{F}_n \text{ for all } n \in \mathbb{N}_*\}.$$

Below S, T are two stopping times. Show that

a) $\mathscr{F}_T$ is a σ-algebra.

b) If $S \leqslant T$ then $\mathscr{F}_S \subseteq \mathscr{F}_T$.

c) $\mathscr{F}_{S\wedge T} = \mathscr{F}_T \cap \mathscr{F}_S$.

d) If $A \in \mathscr{F}_{T\vee S}$ then $A \cap \{T \leqslant S\} \in \mathscr{F}_S$.

e) $\mathscr{F}_{T\vee S} = \sigma(\mathscr{F}_T, \mathscr{F}_S)$.

f) If $(X_n)_n$ is an $(\mathscr{F}_n)_n$-adapted process, then $X_T \,\hat{\in}\, \mathscr{F}_T$.

g) $Z \,\hat{\in}\, \mathscr{F}_T$ if and only if the process $(Z_n = Z \mathbb{1}_{\{T=n\}})_{n\in N_*}$ is adapted, moreover we have $Z_T = Z$.