Institute for Applied Mathematics – SS19
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V3F1 Stochastic Processes – Problem Sheet 5

Distributed May 3rd, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday May 9th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [1+1+1 Pts]

a) Let $(X_n)_{n\geqslant 0}$ and $(Y_n)_{n\geqslant 0}$ two super-martingales and T a stopping time such that $T<\infty \Rightarrow X_T \geqslant Y_T$. Show that the process $(Z_n)_{n\geqslant 0}$ defined by $Z_n = X_n \mathbb{1}_{n\leqslant T} + Y_n \mathbb{1}_{n>T}$ is a super-martingale.

b) Let $(X_n)_{n\geqslant 0}$ a super-martingale such that $\mathbb{E}[X_n]=1$ for all $n\geqslant 0$. Show that it is a martingale.

c) Let $(X_n)_{n\geqslant 0}$ a positive super-martingale, $T=\inf\{n\geqslant 0: X_n=0\}$ and assume that $T<\infty$ almost surely. Show that $X_{T+n}=0$ a.s. for all $n\geqslant 0$. Namely, a positive super-martingale which touches zero, stays there.

Exercise 2. (Lundberg's inequality) [1+1+3 Pts] Let $(X_n)_{n\geqslant 1}$ an i.i.d. sequence. Assume there exists $R>0$ such that $\mathbb{E}[e^{RX_1}]=1$. Let $S_n = X_1+\cdots+X_n$ for $n\geqslant 1$. We want to show that for all $\ell \geqslant 0$ we have

$$\mathbb{P}\left(\max_{1\leqslant k\leqslant n} S_k \geqslant \ell\right) \leqslant e^{-R\ell}.$$

a) Show that the process $(M_n)_{n\geqslant 0}$ given by $M_n = e^{RS_n}$ for all $n\geqslant 1$ and $M_0=1$ is a martingale wrt. the natural filtration of X, denoted here $(\mathscr{F}_n)_{n\geqslant 1}$ with $\mathscr{F}_0=\{\emptyset,\Omega\}$.

b) Show that $T=\inf\{n\geqslant 0: S_n\geqslant \ell\}$ is a stopping time.

c) Show that $\mathbb{P}(\max_{1\leqslant k\leqslant n} S_k \geqslant \ell) \leqslant \mathbb{E}[M_{n\wedge T}]e^{-R\ell}$ and conclude.

Exercise 3. [1+2+2+3 Pts] Let $(X_n)_{n\geqslant 1}$ be an i.i.d. sequence with $\mathbb{P}(X_n=+1)=p\in(0,1)$, $\mathbb{P}(X_n=-1)=q=1-p$. Let $(\mathscr{F}_n)_{n\geqslant 0}$ the natural filtration of X with $\mathscr{F}_0=\{\emptyset,\Omega\}$. Fix $N\geqslant 2$ and let $x\in\{0,1,\ldots,N\}$. Let $S_n = x+X_1+\cdots+X_n$ and $T=\inf\{n\geqslant 0: S_n=0 \text{ or } S_n=N\}$.

a) Let $n\geqslant 0$, show that if $n<T$ and $X_{n+1}=\cdots=X_{n+N-1}=1$ then $T<n+N$.

b) Deduce that $\mathbb{P}(n+N-1<T)\leqslant(1-p^{N-1})\mathbb{P}(n<T)$ and that $T<\infty$ almost surely.

c) Assume $p=q=1/2$. Show that $(S_n)_n$ is a martingale and determine $\mathbb{P}(S_T=0)$ using the optional stopping theorem.

d) Assume now $p\neq q$. Let $M_n=(p/q)^{S_n}$ and show that $(M_n)_n$ is a martingale and determine $\mathbb{P}(S_T=0)$.

Exercise 4. [1+1+2 Pts] Let G a geometric random variable such that $\mathbb{P}(G=k)=p^k(1-p)$ for all $k\geqslant 0$. Let $\mathscr{F}_n=\sigma(G\wedge(n+1))$. Show that

a) $(\mathscr{F}_n)_{n\geqslant 0}$ is a filtration;

b) $(M_n=\mathbb{1}_{G\leqslant n}-(1-p)(G\wedge n))_{n\geqslant 0}$ is a martingale for the filtration $(\mathscr{F}_n)_{n\geqslant 0}$;

c) $(Y_n=M_n^2-p(1-p)(G\wedge n))_{n\geqslant 0}$ is another martingale for the same filtration.

Die Fachschaft Mathematik feiert am 09.05. ihre Matheparty in der N8schicht. Der VVK findet am Mo. 06.05., Di. 07.05. und Mi 08.05. in der Mensa Poppelsdorf statt. Alle weitere Infos auch auf fsmath.uni-bonn.de | The student council of mathematics will organize the math party on 9/05 in N8schicht. The presale will be held on Mon 6/05, Tue 7/05 and Wed 8/05 in the mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de"