Institute for Applied Mathematics – SS19
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V3F1 Stochastic Processes – Problem Sheet 6

Distributed May 10th, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday May 16th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 2+2+2+4] Let $(X_n)_{n\geqslant 1}$ an i.i.d. sequence uniformly distributed on the alphabet $\mathcal{A} = \{A, B, C, \ldots, Z\}$ (with $\#\mathcal{A} = 26$). Let $(\mathcal{F}_n)_{n\geqslant 0}$ the natural filtration of X with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let T_{AB} the first time we observe the sequence AB, namely $T_{\mathrm{AB}} = \inf\{n \geqslant 2 : X_n = B, X_{n-1} = A\}$.

a) Let $Y_n = \sum_{k=2}^{n} 26^2 \mathbb{1}_{X_k = B, X_{k-1} = A} + 26 \mathbb{1}_{X_n = A}$ for $n \geqslant 1$. Show that $M_n = Y_n - n$ is a martingale.

b) Show that $\mathbb{E}[T_{\mathrm{AB}}] = \mathbb{E}[Y_{T_{\mathrm{AB}}}] = 26^2$. (Use optional stopping for $(M_n)_n$)

c) Now show that $\mathbb{E}[T_{\mathrm{BB}}] = 26^2 + 26$.

d) And finally that $\mathbb{E}[T_{\mathrm{ABRACADABRA}}] = 26^{11} + 26^4 + 26$. $T_{\mathrm{ABRACADABRA}}$ is the first time we see the sequence “ABRACADABRA”.

Exercise 2. [Pts 2+2] Let $(X_n)_{n\geqslant 0}$ be a martingale. Show that these two statements are equivalent:

a) There exists positive martingales $(X_n^+)_{n\geqslant 0}$ and $(X_n^-)_{n\geqslant 0}$ such that $X_n = X_n^+ - X_n^-$;

b) X is bounded in L^1.

(Hint: consider $\lim_n \mathbb{E}[X_{m+n}^+ | \mathcal{F}_m]$ for $m \geqslant 0$)

Exercise 3. [Pts 3+3] (Robbins–Monroe algorithm) Let $(X_n)_{n\geqslant 1}$ be an i.i.d. sequence with repartition function $F(t) = \mathbb{P}(X_1 \leqslant t)$ and let $(\mathcal{F}_n)_{n\geqslant 0}$ the natural filtration of X with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We will assume that F is continuous, stricly increasing and for all $\alpha \in (0, 1)$ we let q_α the unique solution to $F(q_\alpha) = \alpha$ (the α-th quantile of F). Let $(Y_n)_{n\geqslant 1}$ the sequence defined by induction via

$$Y_{n+1} = Y_n - \gamma_n (\mathbb{1}_{X_{n+1} \leqslant Y_n} - \alpha), \qquad n \geqslant 0, \tag{1}$$

with Y_0 a fixed, arbitrary constant and $\alpha \in (0, 1)$. The sequence $(\gamma_n)_{n\geqslant 0}$ is positive and decreasing and such that $\sum_n \gamma_n^2 < \infty$, $\sum \gamma_n = +\infty$. The recurrence (1) is a statistical algorithm to approximate the α-th quantile q_α via observations involving only the random variables $(\mathbb{1}_{X_n \leqslant \ell_n})_n$ for a sequence of random levels $(\ell_n)_n$. It is called the Robbins-Monroe algorithm. We want to show that $Y_n \to q_\alpha$ almost surely.

a) Let $(Z_n)_n$ the sequence defined by $Z_n = (Y_n - q_\alpha)^2$. Compute $\mathbb{E}[Z_{n+1} | \mathcal{F}_n]$ and show that there exists an increasing and bounded sequence $(U_n)_{n\geqslant 1}$ such that $W_n = Z_n - U_n$ satisfy

$$0 \leqslant \gamma_n (Y_n - q_\alpha)(F(Y_n) - \alpha) \leqslant W_n - \mathbb{E}[W_{n+1} | \mathcal{F}_n].$$

b) Show that $(W_n)_n$ converges almost surely and that the series

$$\sum_n \gamma_n (Y_n - q_\alpha)(F(Y_n) - \alpha)$$

converges in L^1 and almost surely, and that from this we can deduce $Y_n \to q_\alpha$ almost surely.